Chiral Random Matrix Model as a simple model for QCD

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· HF, T. Sano, Phys. Rev. D 81, 037502; D 83, 014005, and in progress
QCD phase diagram

One of the fundamental challenges in modern physics

Needs non-perturbative analyses

- Lattice QCD at small $\mu$; model studies w/ (P)NJL, etc.
- Beam Energy Scan programs are underway at RHIC/SPS
Chiral Random Matrix (ChRM) model and $U_A(1)$ anomaly

Before we started our project,

**Hope:**
- The simplest model for dynamical breaking of chiral symmetry should reveal the most common features of chiral phase transition

**Problem:**
- It was unknown how to implement $U_A(1)$ breaking term, and then no flavor dependence in ChRM models
Outline

- Motivation
- Chiral Random Matrix (ChRM)
- Incorporating the UA(1) anomaly term
- Meson masses
- Phase diagram – Columbia plot
- (Meson condensation at finite mu at T=0)
- Outlook & Summary
1. Chiral Random Matrix
QCD & Chiral Random Matrix Theory

Review: Verbaarschot-Wettig

- **QCD partition function**

\[
Z^{\text{QCD}} = \int D A_\mu \prod_{f=1}^{N_f} \det(\mathcal{D} + m_f) e^{-S_{\text{YM}}}, \quad \mathcal{D} = \gamma_\mu (\partial_\mu + i A_\mu)
\]

- **Chiral symmetry** \( \{\gamma_5, \mathcal{D}\} = 0 \).

\[
\mathcal{D} \psi_n = i \lambda_n \psi_n, \quad \text{in pair or } \lambda_n = 0; \ n \text{ right-}, m \text{ left-handed modes}
\]

\[
\mathcal{D} = \begin{pmatrix}
0 & i W \\
i W^\dagger & 0
\end{pmatrix}
\]

hermitian with \( W \) is \( n \times m \) complex matrix

\( \mathcal{D} \) has \( \nu = n - m \) exact zero modes (index theorem)

- **Topological sectors**

\[
Z^{\text{QCD}}(\theta) = \sum_{\nu = -\infty}^{\infty} e^{i \nu \theta} Z^{\text{QCD}}_\nu
\]

\[
Z^{\text{QCD}}_\nu = \left\langle \prod_f \hat{m}_{f}^{\nu} \prod_{\lambda_n > 0} (\lambda_n^2 + m_f m^*_f) \right\rangle_\nu
\]

Fluctuation of \( \nu = \) susceptibility w.r.t. \( \theta \)
QCD & Chiral Random Matrix Theory

• Symmetry breaking: Banks-Casher rel

\[
\langle \bar{\psi} \psi \rangle = -\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V N_f} \frac{\partial}{\partial m} \log Z^{QCD}(m) \\
= -\lim_{m \to 0} \lim_{V \to \infty} \left\langle \frac{1}{V} \sum_n \frac{1}{i \lambda_n + m} \right\rangle = \frac{-\pi \rho(0)}{V}
\]

Free theory: \( \lambda = k \), \( \rho(\lambda) \sim \lambda^3 \)

• \( \chi_{SB} \) = accumulation of low-lying Dirac modes by non-perturbative effects

\[
\mathcal{D} = \gamma_\mu (\partial_\mu + i A_\mu) \\
\rho(\lambda) = \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle
\]
QCD & Chiral Random Matrix Theory

Restrict only to low-lying (constant) modes represented by a $2N \times 2N$ matrix with Gaussian random elements (C.f. Nuclear Structure)

QCD partition func. $\rightarrow$ ChRM theory

$$Z^{QCD} = \int D A_\mu \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}}$$

$$Z^{RM} = \int dW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N \Sigma^2 \text{tr} W W^\dagger}$$

Equivalent to QCD in the $\epsilon$ regime, $m_\pi << 1/L << m_\rho$, where constant pion fluctuations dominate in the partition function

Application:
(1) Universal spectral correlation (2) QCD-like model at $N \rightarrow \infty$
Model of QCD: sigma model representation

Shuryak & Verbaarschot (1993)

Integration over $W \rightarrow \int d\bar{\psi}d\psi e^{-\frac{1}{N\Sigma^2} \psi^\dagger_L \psi^g_R \psi^g_R \psi^f_L - m\bar{\psi}\psi}$

Bosonization $\rightarrow \langle \bar{\psi}^f \psi^g \rangle \sim S = \text{diag}(\phi, \ldots, \phi)$

$$Z^{\text{RM}} = \int dS e^{-N\Sigma^2 \text{tr} S^\dagger S} \text{det}^N [(S + m)(S^\dagger + m)] \equiv \int dS e^{-2N\Omega(S)}$$

In thermodynamic limit & equal mass case

$$\Omega(\phi, m) = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log(\phi + m)^2 \right]$$

$$\langle \bar{\psi}\psi \rangle = -\frac{1}{2NN_f} \frac{\partial}{\partial m} \ln Z(m) = -\Sigma^2 \phi.$$  

• Chiral symmetry is broken 🙄

• Nf is factorized 😞

(angular fluctuation of $S$ is equiv to Nonlin sigma model)
Finite Temperature extension

Jackson & Verbaarschot (1996)
Stephanov (1996)

Introduce a deterministic field “t” respecting symmetry

\[ D = \begin{pmatrix} 0 & iW + it \\ iW\dagger + it & 0 \end{pmatrix} \]

\[ \Omega = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log\{(\phi + m)^2 + t^2\} \right] \]

Symmetry restoration at finite T

2\textsuperscript{nd} order for any N\textsubscript{f} (Landau mean-field theory)
Extension to Finite $T$ & $\mu$

$t$ : respects symmetry  $\mu$ : breaks hermiticity

\[ D = \begin{pmatrix} 0 & iW + iC \\ iW^\dagger + iC^T & 0 \end{pmatrix} \]
\[ iC = \begin{pmatrix} (\mu + iT) \times 1_{N/2} & 0 \\ 0 & (\mu - iT) \times 1_{N/2} \end{pmatrix} \]

Consistent with Landau-Ginzburg analysis  $T$& $\mu$ enter by symmetry consideration

Independent of $N_f$
2. Implementing the UA(1) anomaly term
Index theorem in ChRM model

- Index theorem: \[ \nu \equiv N_+ - N_- \]
  - \( N_+ \) - \#(eigenmodes), \( \nu \) : topological \# of a gauge config
  - Instantons \( \rightarrow \) UA(1) breaking

- Total partition fn is obtained by summing over \( \nu \) (w/angle \( \theta \))

\[
Z^{\text{QCD}}(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z_{\nu}^{\text{QCD}}.
\]

- In ChRM model, \( D \) of \( N \times (N + \nu) \) matrix \( W \) has \( \nu \) exact-zero eigenvalues \( (P(\nu): \text{gauge field weight}) \)

\[
D = \begin{pmatrix}
0 & iW \\
-iW^\dagger & 0
\end{pmatrix}
\]

\[
Z^{\text{eff}}(\theta, m) = \sum_{\nu} P(\nu)e^{i\nu\theta} Z_{\nu}^{\text{eff}}(m)
\]
Extension of Zero-mode Space

Janik, Nowak & Zahed (1997)
Sano, HF, Ohtani (2009)

Idea: Divide low-lying modes into two categories

- \( N^+, N^- \): Topological (instanton-) zero modes and fluctuating
- \( 2N \): Near-zero modes

\[
W \in \mathbb{C}^{(N+N_+) \times (N+N_-)}
\]

\[
D = \begin{pmatrix}
0 & iW + iC \\
-iW^T + iC^T & 0
\end{pmatrix}
\]

\[
iC = \begin{pmatrix}
(\mu + iT) \times 1_{N/2} & 0 \\
0 & (\mu - iT) \times 1_{N/2}
\end{pmatrix}
\]

\[
Z_{N^+N^-} = \int dS \ e^{-N \Sigma^2 \text{tr} S^\dagger S} \det \frac{N}{2} \left[ (S + m)(S^\dagger + m^\dagger) - (\mu + iT)^2 \right] \det \frac{N}{2} [C.C.] \\
\times \det^{N_+} (S + m) \det^{N_-} (S^\dagger + m^\dagger)
\]

Last term gives the phase \( e^{2iNf \nu \theta} \) when \( S \to S e^{2i\theta} \)
Complete partition fn. – Sum over \( \nu \) (I)

Janik, Nowak & Zahed (1997)

\[
Z^{\text{RM}}(\theta) = \sum_{N_+,N_-} e^{i\nu\theta} P(N_+) P(N_-) Z^{N_+} Z^{N_-}
\]

\[
= \int dS \sum_{N_+,N_-} e^{i\nu\theta} P(N_+) P(N_-) \cdots \det^{N_+} (S + m) \det^{N_-} (S^\dagger + m^\dagger)
\]

Poisson dist for “instantons”: \( P_{\text{Po}}(N_\pm) = \frac{(gN)^{N_\pm}}{N_\pm!} \)

\[
\Omega_{\text{Po}} = \frac{1}{2} \Sigma^2 \text{tr} S^\dagger S - \frac{1}{2} \log \det \left[ (S + m)(S^\dagger + m^\dagger) + t^2 \right]
\]

\[
- \frac{g}{2} \left[ e^{i\theta} \det (S + m) + e^{-i\theta} \det (S^\dagger + m^\dagger) \right]
\]

KMT-type UA(1) breaking term appears!

Potential is unbound– \( \phi^3 \) term wins at large \( \phi \)

\( P_{\text{Po}} \) dist modified by quark d.o.f.

Note \( \langle N_+ \rangle \propto gN \det (S + m) \sim gN\phi \)

\( \langle N_\pm \rangle \to \infty \) as \( \phi \to \infty \)
Complete partition fn. – Sum over \( \nu \) (II)


Total number of modes must be finite \( N \sim V \)

**Binomial dist**  \[
P(N_{\pm}) = \binom{\gamma N}{N_{\pm}} p^{N_{\pm}} (1 - p)^{\gamma N - N_{\pm}}
\]

\[
\Omega = \frac{1}{2} \sum \text{tr} S^\dagger S - \frac{1}{2} \log \det [(S + m)(S^\dagger + m^\dagger) + t^2] \\
- \frac{\gamma}{2} \left[ \log(\alpha e^{i\theta} \det(S + m) + 1) + \log(\alpha e^{-i\theta} \det(S^\dagger + m^\dagger) + 1) \right]
\]

Finite d.o.f. \( \rightarrow \) \( Z \) is a polynomial (except for Gauss weight)

KMT int. appears under the log. in \( \Omega \)

Stable ground state
Nf Dependent Thermal Phase Transition

\[
m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi)
\]

Chiral condensate

\[\Sigma=1, \alpha=0.3, \gamma=2\]

Nf=2

Nf=3
Topological susceptibility at finite $T$

$$Z^{RM}(\theta) = \sum_{N_+, N_-} P(N_+)P(N_-)e^{i\theta \nu} Z_{N_+, N_-}$$

$$\chi_{\text{top}} \equiv -\frac{1}{2N} \frac{\partial^2}{\partial \theta^2} \ln Z_{\theta} \bigg|_{\theta=0}$$

Unphysical Suppression of $\chi_{\text{top}}$

Adopted from Ohtani’s slide (2007)

ChRM model

Ohtani, Lehner, Wettig & Hatsuda (2008)

B. Alles, M. D’Elia & A. Di Giacomo
NPB483 (2000) 139

as $N \to \infty$
Topological susceptibility at finite $T$

Our model satisfies the UA(1) identity!

$\rightarrow$ consistent with symmetries of QCD

$$\chi_{\text{top}} = -\frac{m^2}{N_f} \chi_{\text{ps}} - \frac{m}{N_f} \langle \bar{\psi} \psi \rangle$$

Top.suscept. follows the chiral condensate for small $m$

$\Sigma=1$, $\alpha=0.3$, $\gamma=2$
Topological susceptibility at finite $T$

$$m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi)$$

$$\Sigma = 1, \alpha = 0.3, \gamma = 2$$

$$\chi_{\text{top}} \equiv -\left. \frac{1}{2N} \frac{\partial^2}{\partial \theta^2} \ln Z_\theta \right|_{\theta = 0}$$

follows the chiral condensate

$$\chi_{\text{top}} = -\frac{m^2}{N_f} \chi_{\text{ps}} - \frac{m}{N_f} \langle \bar{\psi} \psi \rangle$$
3. meson masses
Meson curvature masses

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

\[ M_{ab}^{s^2} = \left. \frac{\partial^2 \Omega(S)}{\partial \sigma_a \partial \sigma_b} \right|_{S=S}, \quad M_{ab}^{ps^2} = \left. \frac{\partial^2 \Omega(S)}{\partial \pi_a \partial \pi_b} \right|_{S=S} \]

\[ (\phi + m)^2 M_{ps}^2 = m \sum^2 (\phi + m) \sim -m \langle \bar{\psi} \psi \rangle \]

\[ M_{s0}^2 = M_s^2 - \Delta M_0^2 \]

\[ M_{ps0}^2 = M_{ps}^2 + \Delta M_0^2 \]

Singlet pseudo-scalar meson is massive by anomaly.
Meson curvature masses

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

\[ M_{ab}^2 = \frac{\partial^2 \Omega(S)}{\partial \sigma_a \partial \sigma_b} \bigg|_{S=S} \quad M_{ab}^{ps2} = \frac{\partial^2 \Omega(S)}{\partial \pi_a \partial \pi_b} \bigg|_{S=S} \]

\[(\phi + m)^2 M_{ps}^2 = m \Sigma^2 (\phi + m) \sim -m \langle \bar{\psi} \psi \rangle \]

\[ M_{s0}^2 = M_s^2 - \Delta M_0^2 \]

\[ M_{ps0}^2 = M_{ps}^2 + \Delta M_0^2 \]

Singlet pseudo-scalar meson is massive by anomaly

All the masses degenerate in symm phase in spite of UA(1) breaking

See Hatsuda-Lee, also Jido's lecture
Singularity at the critical point

\[ m_{ud} = 0.01 \& m_s = 0.2; \ \alpha = 0.5, \gamma = 1 \]

Only \( \sigma \) becomes “massless”

Note that, at CP, \( \sigma \) mixes with density and heat fluctuations; all susceptibilities \( \chi_{\mu\mu}, \chi_{\mu\bar{\mu}}, \chi_{TT} \) diverge
4. Columbia Plot
2+1 flavor phase diagram: $\mu=0$ plane

$$m = \text{diag}(m_{ud}, m_{ud}, m_s)$$

The stronger KMT term makes the 1\textsuperscript{st} order region wider.

Boundary curve is consistent with mean-field prediction.
Critical Surface

1st order region expands as $\mu$ increases

Familiar situation with constant KMT coupling $\alpha=0.5 \ & \gamma=1$

$O(4)$ criticality
5. Meson condensation at $T=0$ at finite $\mu$
Meson condensation at $T=0$, finite $\mu_i$

In vacuum, chiral & meson condensed phases degenerate if $m=0$.
At larger $\mu_i$, a pion condensed phase appears.
At small $\mu_i$, a single chiral transition along $\mu_q$ due to anomaly.


$m = 0.1 \quad \alpha = 0.5 \quad \gamma = 1$

Chemical potentials, and condensates

$$
\mu_q = \frac{1}{2}(\mu_u + \mu_d)
$$

$$
\mu_I = \frac{1}{2}(\mu_u - \mu_d)
$$

$$
S = \begin{pmatrix}
\phi_u \\
\rho_1 - \rho_2 \\
\phi_d
\end{pmatrix}
$$

Gap eq for $\rho$ (m=0)

$$
\rho^2 = \mu_u \mu_d + \frac{1}{\Sigma^2} \left( 1 - \frac{\alpha \gamma}{\alpha \rho^2 + 1} \right)^{-1}
$$
Meson condensation at $T=0$, finite $\mu_I$ & $\mu_Y$

\[
\begin{align*}
\mu_q &= \frac{1}{2}(\mu_u + \mu_d), \\
\mu_I &= \frac{1}{2}(\mu_u - \mu_d), \\
\mu_Y &= \frac{1}{2}(\mu_u + \mu_d - 2\mu_s).
\end{align*}
\]

Kaon condensation appear in the diagram

Chiral restoration & meson conds compete with each other

Regarding CSC phase, see Sano-Yamazaki (2011)
6. Outlook
Complex Langevin simulation
ChRM to study Complex Langevin simulation

$D$ is non(anti)hermitian at finite $\mu$; the same sign problem as QCD → invalidates importance sampling

$$Z^{\text{RM}} = \int dW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N\Sigma^2 \text{tr} W^\dagger W}$$

Langevin eq makes a system distributed around a minimum

$$\frac{\partial \phi}{\partial \tau} = -\Gamma \frac{\partial S}{\partial \phi} + \eta$$

When $S$ becomes complex at some config, originally real vars become complex after evolution (average must be real, though)

Known old problems in Complex Langevin simulations:

- Convergence: avoided using adoptive step size!
- Correctness of equilib dist: trial&error situation
ChRM to study Complex Langevin simulation

Converging, but wrong – sign problem or other reason?

$m=0.4$

N=2; finite system

Sano-HF-Kikukawa, in progress

N=infinity

Han-Stephanov (2008)
Summary

- ChRM model with UA(1) anomaly is constructed consistent with symmetries of QCD
- Fluctuations of #(zero modes) N+- result in “physical” behavior of top. susceptibility
- Meson “masses” and T-μ phase diagram are qualitatively the same as those in other models
- Meson condensation is studied as a response to chemical potentials
- Chiral Random Matrix model is a useful toy model for QCD – e.g., investigation of the sign problem
Introduction: Chiral Random Matrix Theory

- Chiral random matrix theory
  1. Exact description for QCD in $\epsilon$ regime $1/\Lambda_{\text{QCD}} \ll L \ll 1/m_\pi$
  2. A schematic model with chiral symmetry

- In-medium Models
  - Chiral restoration at finite $T$
  - Phase diagram in $T$-$\mu$
  - Sign problem, etc...
  - $U(1)$ problem & resolution (vacuum)


- Known problems at finite $T$
  1. Phase transition is 2nd-order irrespective of $N_f$
  2. Topological susceptibility behaves unphysically

Jackson & Verbaarschot (1996)
Wettig, Schaefer & Weidenmueler (1996)
Halasz et. al. (1998)
Han & Stephanov (2008)
Bloch, & Wettig (2008)
Janik, Nowak, Papp, & Zahed (1997)

Ohtani, Lehner, Wettig & Hatsuda (2008)