QCD phase transition

- One of the fundamental challenges in modern physics
- Non-perturbative phenomena
- No theoretical control at $\mu > 0$ so far...
Chiral Random Matrix (ChRM) model and $U_A(1)$ anomaly

Before we started our study ..., 

Hope:

● The simplest model for dynamical breaking of chiral symmetry – no spacetime D.O.F. – may reveal the most common features of chiral phase transition

Problem:

● In mean-field models, the KMT term is responsible for $U_A(1)$ breaking; But it was unknown how to implement it, and then no flavor dependence
Outline

• Introduction

• Chiral Random Matrix (ChRM) Model in brief

• Incorporating the UA(1) anomaly term

• Phase diagram at finite $T/\mu$

• Meson condensation at finite $\mu$ at $T=0$

• Conclusions & Outlook
2. Chiral Random Matrix Model in brief
QCD & Chiral Random Matrix Theory

- The QCD partition function
  \[ Z^{QCD} = \int D A_\mu \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}}, \]

- Chiral symmetry breaking ($\chi_{SB}$)
  \[
  \{\gamma_5, D\} = 0. \quad D \psi_n = i \lambda_n \psi_n,
  \]
  \[
  \langle \overline{\psi} \psi \rangle = -\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V N_f} \frac{\partial}{\partial m} \log Z^{QCD}(m)
  \]
  \[= -\lim_{m \to 0} \lim_{V \to \infty} \left\langle \frac{1}{V} \sum_n \frac{1}{i \lambda_n + m} \right\rangle = -\pi \rho(0) \frac{V}{V} \]

- $\chi_{SB}$ is characterized by low-lying Dirac modes, which are generated by complex YM dynamics
QCD & Chiral Random Matrix Theory

- The QCD partition function $Z^{QCD}$ → ChRM theory

$$Z^{QCD} = \int D A_{\mu} \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}} \quad \Rightarrow \quad Z^{RM} = \int dW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N \Sigma^2 \text{tr} W W^\dagger}$$

Dirac operator matrix with constant elements distributed Gaussian-randomly

$$\{\gamma_5, D\} = 0, \quad D = \begin{pmatrix} 0 & i W \\ i W^\dagger & 0 \end{pmatrix}$$

Equivalence to QCD in the $\varepsilon$ regime is known: $m_\pi << 1/L << m_\rho$

Size of matrix ($2N \times 2N$) is proportional to box size, $L^4$

Application:

1. Universal spectral correlation
2. A model for QCD at $N \to \infty$
Model of QCD: effective Potential $\Omega$

Integration over $W \rightarrow \int d\overline{\psi}d\psi e^{-\frac{1}{N\Sigma^2}\psi^+_{L}^{g}\psi_{R}^{g}\psi^+_R^{g}\psi_L^{g}-m\overline{\psi}\psi}$

Bosonization $\rightarrow \langle \overline{\psi}^{f}\psi^{g}\rangle \sim S = \text{diag}(\phi, \ldots, \phi)$

$$Z^{RM} = \int dSe^{-N\Sigma^2\text{tr}S\dagger S} \det^N[(S + m)(S\dagger + m)] \equiv \int dSe^{-2N\Omega(S)}$$

In thermodynamic limit & equal mass case

$$\Omega(\phi, m) = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log(\phi + m)^2 \right]$$

- Chirally broken ground state
- $N_f$ is factorized

Shuryak & Verbaarschot (1993)
Finite Temperature extension

Jackson & Verbaarschot (1996)

Introduce a non-random external field $\mathbf{t}$ respecting symmetry

$$D = \begin{pmatrix} 0 & iW + it \\ iW^\dagger + it & 0 \end{pmatrix}$$

effective potential

$$\Omega = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log\{(\phi + m)^2 + t^2\} \right]$$

Symmetry restoration at finite $T$

2$^{nd}$ order for any $N_f$

- “Determinant” int is needed to study flavor dependence
Extension to Finite T & \( \mu \)

- \( t \) : respects symmetry
- \( \mu \) : breaks hermiticity

\[ D = \begin{pmatrix} 0 & iW + iC \\ iW^\dagger + iC^T & 0 \end{pmatrix} \quad W \in \mathbb{C}^{N \times N} \]

\[ iC = \begin{pmatrix} (\mu + iT) \times 1_{N/2} & 0 \\ 0 & (\mu - iT) \times 1_{N/2} \end{pmatrix} \]

\[
\Omega = \frac{1}{2} \sum \text{Tr}S^\dagger S - \frac{1}{4} \log \left[ \det ((S + m)(S^\dagger + m^\dagger) - (\mu + iT)^2) \det ((S + m)(S^\dagger + m^\dagger) - (\mu - iT)^2) \right]
\]

\[ S = \text{diag}(\phi, \ldots, \phi) \quad m = \text{diag}(m, \ldots, m) \]

Independent of \( N_f \)

Halasz et al. (1998)
3. Implementing the UA(1) anomaly term
Index theorem in ChRM model

\[
\{\gamma_5, D\} = 0. \quad D\psi_n = i\lambda_n \psi_n,
\]

- non-zero eigenvalues appear in pair with + – chiralities
  - Exception: zero eigenvalues
- Index theorem:
  \[\nu \equiv N_+ - N_-\]
  - \(N_+ -\): #(eigenmodes), \(\nu\): topological # of gauge configuration
  - Total partition fn is obtained by summing over \(\nu\) (w/angle \(\theta\))

\[
Z^{QCD}(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} Z^{QCD}_\nu.
\]

- In ChRM model, we deal with \(N \times (N + \nu)\) matrix \(W\), which has \(\nu\) exact-zero eigenvalues
Extension of Zero-mode Space

Janik, Nowak & Zahed (1997)
Sano, HF, Ohtani (2009)

Divide low-lying modes into two categories

- N+, N- : Topological (quasi-) zero modes = associated with instantons(?)
- 2N : Near zero modes

\[
D = \begin{pmatrix}
0 & iW + iC \\
iW^T + iC^T & 0
\end{pmatrix}
\]

\[
W \in \mathbb{C}^{(N+N_+) \times (N+N_-)}
\]

Near-zero mode

\[
iC = \begin{pmatrix}
(\mu + iT) \times 1_{N/2} & 0 \\
0 & (\mu - iT) \times 1_{N/2}
\end{pmatrix}
\]

\[
Z_{N_+N_-} = \int dS \ e^{-N\Sigma^2 \text{tr}S^\dagger S} \det \frac{N}{2} [(S + m)(S^\dagger + m^\dagger) - (\mu + iT)^2] \det \frac{N}{2} [C.C.]
\times \det^{N_+}(S + m) \det^{N_-}(S^\dagger + m^\dagger)
\]

The last term gives a phase \( e^{2iNf \nu \theta} \) when \( S \rightarrow S \ e^{2i\theta} \)
Complete partition fn. – Sum over $v$ (I)

Janik, Nowak & Zahed (1997)

$$Z^{RM}(\theta) = \sum_{N_+, N_-} P(N_+)P(N_-)e^{i\theta\nu}Z_{N_+,N_-} \equiv \int dS e^{-2N\Omega}$$

If modes are associated to instantons, Poisson dist may be reasonable

$$P_{Po}(N_\pm) = \frac{(gN)^{N_\pm}}{N_\pm!}$$

$$\Omega_{Po} = \frac{1}{2} \Sigma^{2} \text{tr} S^{\dagger} S - \frac{1}{2} \log \det [(S + m)(S^{\dagger} + m^{\dagger}) + t^2]$$

$$- \frac{g}{2} \left[ e^{i\theta} \det(S + m) + e^{-i\theta} \det(S^{\dagger} + m^{\dagger}) \right]$$

KMT-type UA(1) breaking term, appears

Potential is unbound – $\phi^3$ term wins at large $\phi$

Note $\langle N_\pm \rangle \propto gN \det(S + m) \sim gN\phi$
Complete partition fn. – Sum over $\nu$ (II)


- With the Poisson, $\langle N_\pm \rangle \to \infty$ as $\phi \to \infty$
- ??? model restricted in IR modes in finite $V$
- The most natural regularization: Binomial dist

$$P(N_\pm) = \binom{\gamma N}{N_\pm} p^{N_\pm} (1 - p)^{\gamma N - N_\pm}$$

$$\Omega = \frac{1}{2} \Sigma^2 \text{tr} S^\dagger S - \frac{1}{2} \log \det [(S + m)(S^\dagger + m^\dagger) + t^2]$$

$$- \frac{\gamma}{2} \left[ \log(\alpha e^{i\theta} \det(S + m) + 1) + \log(\alpha e^{-i\theta} \det(S^\dagger + m^\dagger) + 1) \right]$$

$Z$ is a polynomial (except for Gauss weight)

KMT int. appears within the log.

Stable ground state
4. Phase diagram at finite $T$ & $\mu$
Nf Dependent Thermal Phase Transition

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

Chiral condensate

\[ \Sigma=1, \, \alpha=0.3, \, \gamma=2 \]
Nf Dependent Thermal Phase Transition

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

Meson (curvature) mass

\[ M_{ab}^{s} = \frac{\partial^{2} \Omega(S)}{\partial \sigma_{a} \partial \sigma_{b}} \bigg|_{S = \bar{S}}, \quad M_{ab}^{ps} = \frac{\partial^{2} \Omega(S)}{\partial \pi_{a} \partial \pi_{b}} \bigg|_{S = \bar{S}} \]

\[ \Sigma = 1, \alpha = 0.3, \gamma = 2 \]

Singlet pseudo-scalar meson is massive by anomaly
Singularity at the critical point

\[ m_{ud}=0.01 \& m_s=0.2; \quad \alpha=0.5, \gamma=1 \]

Only \( \sigma \) becomes "massless"

Cf. Susceptibilities \( \chi_{mm}, \chi_{\mu\mu}, \chi_{TT} \) all diverge at the CP
Topological susceptibility at finite $T$

$$\chi_{\text{top}} \equiv - \frac{1}{2N} \frac{\partial^2}{\partial \theta^2} \ln Z_\theta \bigg|_{\theta=0}$$

**Unphysical Suppression of $\chi_{\text{top}}$**

Adopted from Ohtani’s slide (2007)

ChRM model: Ohtani, Lehner, Wettig & Hatsuda (2008)

Lattice: B. Alles, M. D’Elia & A. Di Giacomo
NPB483 (2000) 139

Our model describes physical $\chi_{\text{top}}$ & Nf-dependent phase transition
Topological susceptibility at finite T

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \quad \Sigma=1, \alpha=0.3, \gamma=2 \]

\[ \chi_{\text{top}} \equiv -\frac{1}{2N} \frac{\partial^2}{\partial \theta^2} \ln Z_\theta \bigg|_{\theta=0} \]

follows the chiral condensate

\[ \chi_{\text{top}} = -\frac{m^2}{N_f} \chi_{\text{ps}} - \frac{m}{N_f} \langle \bar{\psi}\psi \rangle \]
$m = \text{diag}(m_{ud}, m_{ud}, m_s)$

The stronger the KMT term, the wider the 1\textsuperscript{st} order region

Expt 2/5 is understood in the Landau potential
Critical Surface

1st order region expands as μ increases

Familiar situation with constant KMT coupling

\[ \alpha = 0.5 \& \gamma = 1 \]

O(4) criticality
If $\alpha$ depends on $\mu$ ...

\[
\alpha = \alpha_0 \exp \left( -\frac{\mu^2}{\mu_0^2} \right)
\]

$\gamma=1$, $\alpha_0=0.5$, & $\mu_0=0.2$

Bending is possible, but model dependent
5. Meson condensation at $T=0$ at finite $\mu$
Meson condensation at finite $\mu_i$

At large $\mu_i$, a pion condensed phase appears

At small $\mu_i$, a single chiral transition along $\mu_q$

$\mu_q = \frac{1}{2}(\mu_u + \mu_d)$

$\mu_I = \frac{1}{2}(\mu_u - \mu_d)$

$S = \begin{pmatrix} \phi_u & i\rho_1 + \rho_2 \\ i\rho_1 - \rho_2 & \phi_d \end{pmatrix}$

Gap eq for $\rho$ (m=0)

$\rho^2 = \mu_u\mu_d + \frac{1}{\Sigma^2} \left(1 - \frac{\alpha\gamma}{\alpha\rho^2 + 1}\right)^{-1}$

Meson condensation at finite $\mu_I$ & $\mu_Y$

$$\mu_q = \frac{1}{2}(\mu_u + \mu_d),$$
$$\mu_I = \frac{1}{2}(\mu_u - \mu_d),$$
$$\mu_Y = \frac{1}{2}(\mu_u + \mu_d - 2\mu_s).$$

$$S = \begin{pmatrix} \phi_u & \rho_{ud} & -\rho_{su} \\ -\rho_{ud} & \phi_d & \rho_{ds} \\ \rho_{su} & -\rho_{ds} & \phi_s \end{pmatrix}$$

HF, T. Sano,

$$m_u = m_d = 0.02, m_s = 0.1$$

Kaon condensed phases appear in some regions

Competition among chiral restoration & meson condensations
5. Summary

- Chiral Random Matrix model with UA(1) anomaly is constructed as a mean-field model for QCD
- Chiral condensate, meson masses, topological susceptibility behave 'physically' as $T$ & $\mu$ change
- Phase diagram in $T-\mu$ plane is qualitatively the same as other mean-field models
- Meson condensation is studied as a response to chemical potentials
- Chiral Random Matrix model is a useful toy model for QCD