Chiral Random Matrix Model
with $U_A(1)$ Anomaly at finite $T$ and $\mu$

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· HF, T. Sano, Phys. Rev. D 81, 037502; D 83, 014005, and in progress
One of the fundamental challenges in modern physics

Needs non-perturbative analyses

- Lattice QCD at small $\mu$, model studies w/ (P)NJL, etc.
- Beam Energy Scan programs are underway at RHIC/SPS
Chiral Random Matrix (ChRM) model and $U_A(1)$ anomaly

Before we started our project,

Hope:

- The simplest model for dynamical breaking of chiral symmetry should reveal the most common features of chiral phase transition

Problem:

- It was unknown to implement the $U_A(1)$ breaking KMT term, and then no flavor dependence in ChRM models
Outline

- Motivation
- Chiral Random Matrix (ChRM)
- Incorporating the UA(1) anomaly term
- Phase diagram at finite T/\mu
- Meson condensation at finite \mu at T=0
- Conclusions & Outlook
1. Chiral Random Matrix
QCD & Chiral Random Matrix Theory

Review: Verbaarschot-Wettig

- **QCD partition function**
  \[ Z_{\text{QCD}} = \int D\!A_\mu \prod_{f=1}^{N_f} \det(\mathcal{D} + m_f) e^{-S_{\text{YM}}}, \quad \mathcal{D} = \gamma_\mu (\partial_\mu + i A_\mu) \]

- **Chiral symmetry** \[ \{\gamma_5, \mathcal{D}\} = 0. \]
  \[ \mathcal{D}\psi_n = i\lambda_n \psi_n, \quad \text{in pair or } \lambda_n = 0; \ n \text{ right-}, m \text{ left-handed modes} \]
  \[ \mathcal{D} = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix} \quad \text{in chiral basis with } W \text{ is } n \times m \text{ complex matrix} \]
  \[ D \text{ has } v = n - m \text{ exact zero modes (index theorem)} \]

- **Topological sectors**
  \[ Z_{\text{QCD}}(\theta) = \sum_{v=-\infty}^{\infty} e^{i\nu\theta} Z_{v}^{\text{QCD}} \]
  \[ Z_{v}^{\text{QCD}} = \left\langle \prod_{f} \hat{m}_{[v]}^{[f]} \prod_{\lambda_n > 0} (\lambda_n^2 + m_f m_f^*) \right\rangle_v \]
  \[ v \text{ fluctuation = susceptibility w.r.t. } \theta \text{ (which may be absorbed in } m) \]
QCD & Chiral Random Matrix Theory

- Symmetry breaking: Banks-Casher

\[
\langle \bar{\psi} \psi \rangle = -\lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V N_f} \frac{\partial}{\partial m} \log Z^{QCD}(m)
\]

\[
= -\lim_{m \to 0} \lim_{V \to \infty} \left\langle \frac{1}{V} \sum_n \frac{1}{i \lambda_n + m} \right\rangle = \frac{-\pi \rho(0)}{V}
\]

Free theory: \( \lambda = k, \quad \rho(\lambda) \sim \lambda^3 \)

- \( \chi_{SB} = \) accumulation of low-lying Dirac modes due to non-perturbative effects
QCD & Chiral Random Matrix Theory

Restrict only to low-lying (constant) modes represented by a 2Nx2N matrix with Gaussian random elements (C.f. Nuclear Structure)

QCD partition func. $\rightarrow$ ChRM theory

$$Z^{QCD} = \int D A_\mu \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}}$$

$$Z^{RM} = \int dW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N\Sigma^2_{tr} W^\dagger W}$$

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}$$

Equivalent to QCD in the $\varepsilon$ regime, where constant pion fluctuations dominate the partition function: $m_{\pi} << 1/L << m_{\rho}$

Application:
(1) Universal spectral correlation (2) QCD-like model at $N \rightarrow \infty$
Model of QCD: sigma model representation

Shuryak & Verbaarschot (1993)

Integration over $W \to \int d\overline{\psi} d\psi e^{-\frac{1}{N\Sigma^2} \psi^f_L \psi^g_R \psi^g_R \psi^f_L - m\overline{\psi}\psi}$

Bosonization $\to \langle \overline{\psi}^f \psi^g \rangle \sim S = \text{diag}(\phi, \ldots, \phi)$

$$Z^{RM} = \int dS e^{-N\Sigma^2 \text{tr}S^\dagger S} \det^N [(S + m)(S^\dagger + m)] = \int dS e^{-2N\Omega(S)}$$

In thermodynamic limit & equal mass case

$$\Omega(\phi, m) = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log(\phi + m)^2 \right]$$

$$\langle \overline{\psi} \psi \rangle = -\frac{1}{2NN_f} \frac{\partial}{\partial m} \ln Z(m) = -\Sigma^2 \phi.$$  

- Chiral symmetry is broken
- $N_f$ is factorized

(angular fluctuation is equiv to Nonlin sigma model)
Finite Temperature extension

Jackson & Verbaarschot (1996)
Stephanov (1996)

Introduce a deterministic field “t” respecting symmetry

\[
D = \begin{pmatrix}
0 & iW + it \\
iW^\dagger + it & 0
\end{pmatrix}
\]

Effective potential

\[
\Omega = \frac{N_f}{2} \left[ \Sigma^2 \phi^2 - \log \{(\phi + m)^2 + t^2\} \right]
\]

Symmetry restoration at finite T

2\textsuperscript{nd} order for any Nf (Landau mean-field theory)
Extension to Finite $T$ & $\mu$

$D = \begin{pmatrix} 0 & iW + iC \\ iW^\dagger + iC^T & 0 \end{pmatrix}$

$iC = \begin{pmatrix} (\mu + iT) \times 1_{N/2} & 0 \\ 0 & (\mu - iT) \times 1_{N/2} \end{pmatrix}$

$\Omega = \frac{1}{2} \Sigma^2 \Tr S^\dagger S - \frac{1}{4} \log \left[ \det \left( (S + m)(S^\dagger + m^\dagger) - (\mu + iT)^2 \right) \det \left( (S + m)(S^\dagger + m^\dagger) - (\mu - iT)^2 \right) \right]$
2. Implementing the UA(1) anomaly term
Index theorem in ChRM model

\[
\{\gamma_5, D\} = 0.
\]

\[D\psi_n = i\lambda_n\psi_n,\]

- non-zero eigenvalues appear in pair with + – chiralities
  - Exception: zero eigenvalues
- Index theorem: \[\nu \equiv N_+ - N_-\]
  - \(N_+ - \): #(eigenmodes), \(\nu\) : topological # of a gauge configuration
  - Total partition fn is obtained by summing over \(\nu\) (w/angle \(\theta\))

\[Z_{\text{QCD}}(\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta}Z_{\nu}^{\text{QCD}}.\]

- In ChRM model, \(D\) has two blocks of \(N \times (N + \nu)\) matrix \(W\), and has \(\nu\) exact-zero eigenvalues \((P(\nu)\): gauge field weight)
Extension of Zero-mode Space

**Divide low-lying modes into two categories**

- **N+, N-**: Topological (quasi-) zero modes = associated with instantons(?)
- **2N**: Near-zero modes

\[
D = \begin{pmatrix}
0 & iW + iC \\
iW^\dagger + iC^T & 0
\end{pmatrix}
\]

\[
W \in \mathbb{C}^{(N+N_+) \times (N+N_-)}
\]

\[
iC = \begin{pmatrix}
(\mu + iT) \times 1_{N/2} & 0 \\
0 & (\mu - iT) \times 1_{N/2}
\end{pmatrix}
\]

\[
Z_{N+N_-} = \int dS \ e^{-N\Sigma^2 trS^\dagger S} \det^{\frac{N}{2}} \left[ (S + m)(S^\dagger + m^\dagger) - (\mu + iT)^2 \right] \det^{\frac{N}{2}} [C.C.]
\times \det^{N_+} (S + m) \det^{N_-} (S^\dagger + m^\dagger)
\]

The last term gives a phase \( e^{2iNf \nu \theta} \) when \( S \rightarrow S e^{2i\theta} \).
Complete partition fn. – Sum over $\nu$ (I)  

Janik, Nowak & Zahed (1997)

$$Z^{RM} (\theta) = \sum_{N_+, N_-} e^{i\nu \theta} P(N_+) P(N_-) Z_{N_+, N_-}^N$$

$$= \int dS \sum_{N_+, N_-} e^{i\nu \theta} P(N_+) P(N_-) \cdots \det^{N_+}(S + m) \det^{N_-}(S^\dagger + m^\dagger)$$

Poisson dist for “instantons”  

$$P_{Po}(N_\pm) = \frac{(gN)^{N_\pm}}{N_\pm!}$$  

't Hooft (1986)

$$\Omega_{Po} = \frac{1}{2} \Sigma^2 \text{tr} S^\dagger S - \frac{1}{2} \log \det [(S + m)(S^\dagger + m^\dagger) + t^2]$$

$$- \frac{g}{2} \left[ e^{i\theta} \det (S + m) + e^{-i\theta} \det (S^\dagger + m^\dagger) \right]$$

KMT-type UA(1) breaking term appears

Potential is unbound – $\phi^3$ term wins at large $\phi$

Note $\langle N_+ \rangle \propto qN \det (S + m) \sim gN \phi$

$\langle N_\pm \rangle \to \infty$ as $\phi \to \infty$
Complete partition fn. – Sum over $\nu$ (II)

- Model in a finite $V$ has modes whose number $N \sim V$
- Binomial dist

\[
P(N_\pm) = \binom{\gamma N}{N_\pm} p^{N_\pm} (1 - p)^{\gamma N - N_\pm}
\]

\[
\Omega = \frac{1}{2} \sum \text{tr} S^\dagger S - \frac{1}{2} \log \det [(S + m)(S^\dagger + m^\dagger) + t^2] \\
- \frac{\gamma}{2} \left[ \log(\alpha e^{i\theta} \det(S + m) + 1) + \log(\alpha e^{-i\theta} \det(S^\dagger + m^\dagger) + 1) \right]
\]

Finite d.o.f. $\rightarrow Z$ is a polynomial (except for Gauss weight)

KMT int. appears under the log. in $\Omega$

Stable ground state
Topological susceptibility at finite $T$

$$\chi_{\text{top}} \equiv -\frac{1}{2N} \frac{\partial^2}{\partial \theta^2} \ln Z_\theta \bigg|_{\theta=0}$$

Unphysical Suppression of $\chi_{\text{top}}$

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**ChRM model**

Adopted from Ohtani's slide (2007)

$\chi$ vs $T/T_c$

Ohtani, Lehner, Wettig & Hatsuda (2008)

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**lattice**

$\chi(T)/\chi(T=0)$ vs $T/T_c$

B. Alles, M. D'Elia & A. Di Giacomo
NPB483 (2000) 139
Topological susceptibility at finite $T$

Our model satisfies the U(1) identity!

$\to$ consistent with symmetries of QCD

$$\chi_{\text{top}} = -\frac{m^2}{N_f} \chi_{\text{ps}} - \frac{m}{N_f} \langle \bar{\psi}\psi \rangle$$

$\Sigma=1$, $\alpha=0.3$, $\gamma=2$

Top. suscept. follows the chiral condensate
3. Phase diagram at finite $T$ & $\mu$
Nf Dependent Thermal Phase Transition

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

Chiral condensate

\[ \Sigma = 1, \; \alpha = 0.3, \; \gamma = 2 \]

Nf=2

Nf=3
Nf Dependent Thermal Phase Transition

\[ m = \text{diag}(m, \ldots, m) \quad S = \text{diag}(\phi, \ldots, \phi) \]

\( \Sigma = 1, \alpha = 0.3, \gamma = 2 \)

Meson (curvature) mass

\[ M_{ab}^{s^2} = \left. \frac{\partial^2 \Omega(S)}{\partial \sigma_a \partial \sigma_b} \right|_{S = \bar{S}}, \quad M_{ab}^{ps^2} = \left. \frac{\partial^2 \Omega(S)}{\partial \pi_a \partial \pi_b} \right|_{S = \bar{S}} \]

\( N_f = 2 \)

Singlet pseudo-scalar meson is massive by anomaly

\( N_f = 3 \)
Singularity at the critical point

\[ m_{ud} = 0.01 \& m_s = 0.2; \ \alpha = 0.5, \gamma = 1 \]

Only \( \sigma \) becomes “massless”

Note that, at CP, \( \sigma \) mixes with density and heat fluctuations; all susceptibilities \( \chi_{mm}, \chi_{\mu\mu}, \chi_{TT} \) diverge
2+1 flavor phase diagram: $\mu=0$ plane

$$m = \text{diag}(m_{ud}, m_{ud}, m_s)$$

The stronger KMT term makes the 1$^{\text{st}}$ order region wider.

Boundary curve is consistent with mean-field prediction.
Critical Surface

1st order region expands as \( \mu \) increases

Familiar situation with constant KMT coupling \( \alpha=0.5 \) & \( \gamma=1 \)

O(4) criticality
4. Meson condensation at $T=0$ at finite $\mu$
Meson condensation at $T=0$, finite $\mu_I$

Chemical potentials, and condensates

$$\mu_q = \frac{1}{2}(\mu_u + \mu_d)$$
$$\mu_I = \frac{1}{2}(\mu_u - \mu_d)$$

$$S = \begin{pmatrix} \phi_u & i\rho_1 + \rho_2 \\ i\rho_1 - \rho_2 & \phi_d \end{pmatrix}$$

Gap eq for $\rho$ (m=0)

$$\rho^2 = \mu_u\mu_d + \frac{1}{\Sigma^2} \left(1 - \frac{\alpha\gamma}{\alpha\rho^2 + 1}\right)^{-1}$$

In vacuum, chiral & meson condensed phases degenerate if $m=0$
At larger $\mu_I$, a pion condensed phase appears
At small $\mu_I$, a single chiral transition along $\mu_q$ due to anomaly

HF, T. Sano (2010),
Meson condensation at $T=0$, finite $\mu_I$ & $\mu_Y$

$$\mu_q = \frac{1}{2}(\mu_u + \mu_d),$$
$$\mu_I = \frac{1}{2}(\mu_u - \mu_d),$$
$$\mu_Y = \frac{1}{2}(\mu_u + \mu_d - 2\mu_s).$$

$$S = \begin{pmatrix} \phi_u & \rho_{ud} & -\rho_{su} \\ -\rho_{ud} & \phi_d & \rho_{ds} \\ \rho_{su} & -\rho_{ds} & \phi_s \end{pmatrix}$$

HF, T. Sano (2010),

$m_u = m_d = 0.02$, $m_s = 0.1$

Kaon condensation appear in the diagram
Chiral restoration & meson conds compete with each other
Regarding CSC phase, see Sano-Yamazaki (2011)
5. Outlook
ChRM to study Complex Langevin simulation

\( D \) is non(anti)hermitian at finite \( \mu \); the same sign problem as QCD → invalidates importance sampling

\[
Z^{\text{RM}} = \int dW \prod_{f=1}^{N_f} \det(D + m_f) e^{-N \Sigma^2 \text{tr} W^\dagger W}
\]

Langevin eq makes a system distributed around a minimum

\[
\frac{\partial \phi}{\partial \tau} = -\Gamma \frac{\partial S}{\partial \phi} + \eta
\]

When \( S \) becomes complex at some config, originally real vars become complex after evolution (average must be real, though)

Known old problems in Complex Langevin simulations:

- Convergence: avoided using adoptive step size!
- Correctness of equilib dist: trial&error situation
ChRM to study Complex Langevin simulation

Converging, but wrong – sign problem or other reason?

\[ m = 0.4 \]

N=2; finite system

Sano-HF-Kikukawa, in progress

N=\infty

\[ R \equiv \langle e^{2i\theta} \rangle_{1+1^*} \equiv \frac{\langle \det \mathbb{D} \rangle}{\langle \det \mathbb{D}^* \rangle}_{1+1^*} \]

Han-Stephanov (2008)
Summary

• ChRM model with UA(1) anomaly is constructed consistent with symmetries of QCD
• Fluctuations of #(zero modes) N+– result in “physical” behavior of top. susceptibility
• $T-\mu$ phase diagram and meson “masses” are qualitatively the same as those in other models
• Meson condensation is studied as a response to chemical potentials
• Chiral Random Matrix model is a useful toy model for QCD – e.g., investigation of the sign problem