Recent progress in dual superconductor picture for quark confinement

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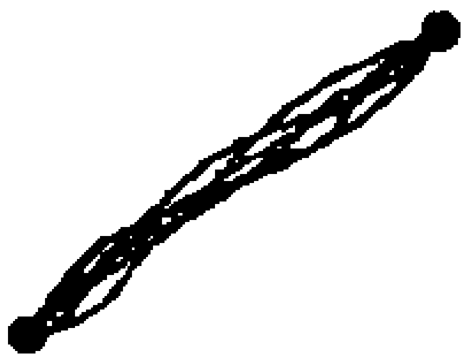
Chapter I
Introduction

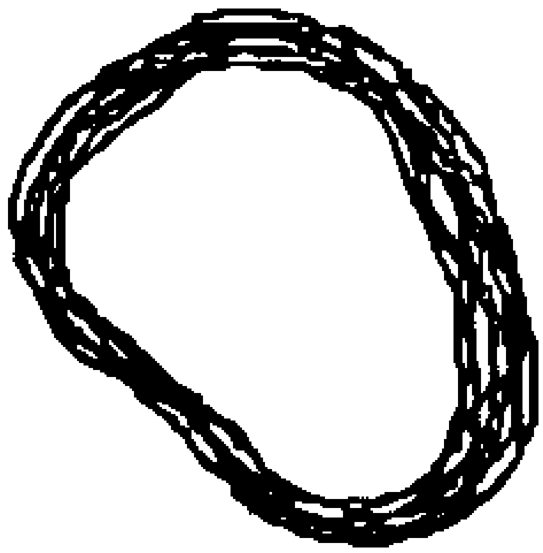
§ Introduction

The fundamental degrees of freedom of QCD, i.e., quarks and gluons, have never been observed in experiments.

Only the color singlet combinations, hadrons (mesons and baryons) and glueballs have been observed so far.

Hadron (Meson, Baryon) and Glueball?





This is color confinement.

Why does color confinement occur?

In particular, what is the mechanism for quark confinement?

- quark-antiquark potential

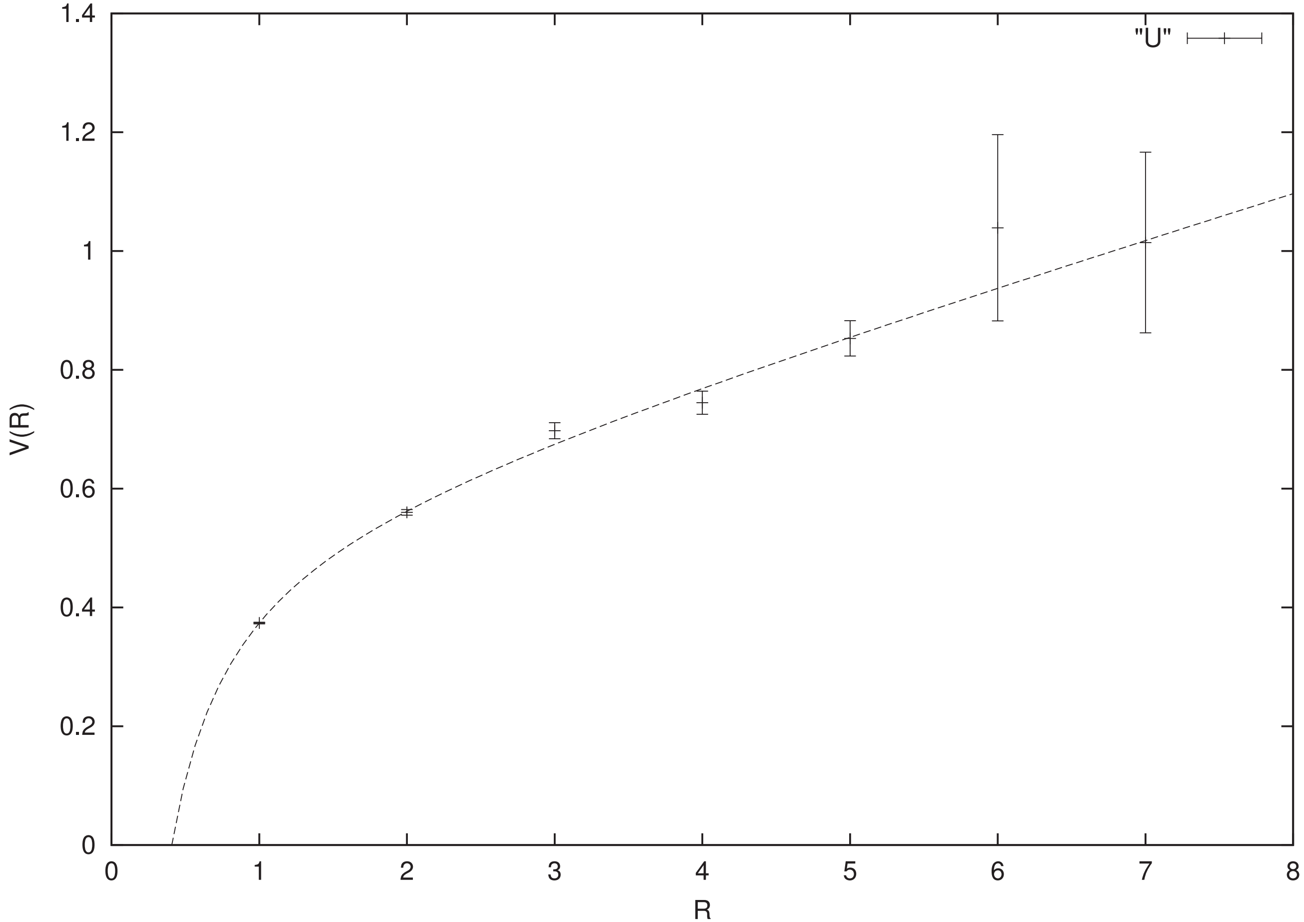
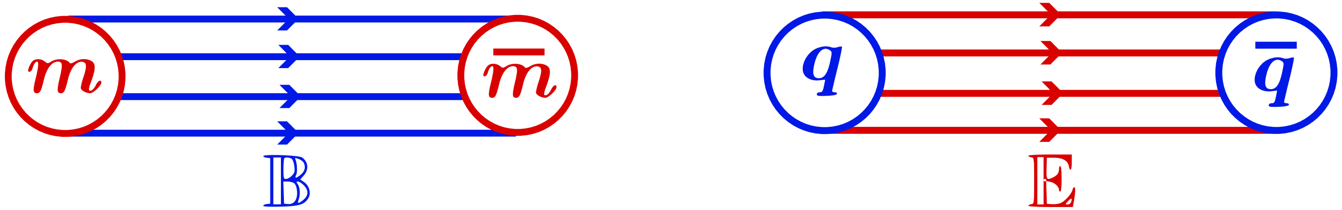


Figure 1: The full SU(2) potential $V_f(R)$ as functions of R at $\beta = 2.4$ on 16^4 lattice.

$$V(r) = -C\frac{g_{\mathrm{YM}}^2(r)}{r} + \sigma r \Longrightarrow F(r) = -\frac{d}{dr}V(r) = -C\frac{g_{\mathrm{YM}}^2(r)}{r^2} - \sigma + \cdots \ (C, \sigma > 0) \quad (1)$$

§ Dual superconductor picture for confinement

[Nambu, 1974] ['tHooft, 1975][Mandelstam, 1976]



(Left panel) superconductor (Right panel) dual superconductor

Superconductivity (type II)
condensation of electric charges (Cooper pairs)
$\Downarrow$
Meissner effect:
formation of Abrikosov string (magnetic flux tube)
connecting a monopole m and an anti-monopole $\bar{m}$
$\Downarrow$
linear potential between a monopole m and an anti-monopole $\bar{m}$

$\Updownarrow$ electric-magnetic dual

linear confining potential between q and $\bar{q}$
$\Uparrow$
formation of a hadron string (electric flux tube)
connecting q and $\bar{q}$
dual Meissner effect:

$\Uparrow$

condensation of **magnetic monopoles**

$\cdot$ Dual superconductivity in YM theory !?

D=2: Yang-Mills theory is exactly calculable, $V(r) = \sigma r$, $\sigma = c_2(N)\frac{g^2}{2} = \frac{N^2-1}{2N}\frac{g^2}{2}$.

Coulomb potential = linear potential in D=2!

Dual superconductor picture was always valid in the following models where confinement was shown in the analytical way.

D=3: • compact QED_3 in Georgi-Glashow model [Polyakov, 1977]

$\longrightarrow$ magnetic monopole plasma, sine-Gordon theory described by the dual variable

D=4: • (Lattice) compact QED_4 (in the strong coupling region) [Polyakov, 1975]

$\longrightarrow$ magnetic monopole plasma ;

U(1) link variable $\rightarrow$ monopole current variable [Banks, Myerson and Kogut, 1977]

• N=2 SUSY YM_4 [Seiberg and Witten, 1994] ...

How about YM_3, YM_4 and QCD_4? Can we introduce magnetic monopoles in these theories? Abelian projection, partial gauge fixing [G. 't Hooft, 1981]

§ Abelian projection and magnetic monopole

Consider the (pure) Yang-Mills theory with the gauge group $G = SU(N)$ on $\mathbb{R}^D$.

(1) Let $\chi(x)$ be a Lie-algebra $\mathscr{G}$-valued functional of the Yang-Mills field $\mathscr{A}_\mu(x)$. Suppose that it transforms in the adjoint representation under the gauge transformation:

$$\chi(x) \to \chi'(x) := U(x)\chi(x)U^\dagger(x) \in \mathscr{G} = su(N), \quad U(x) \in G, \quad x \in \mathbb{R}^D. \tag{1}$$

(2) Diagonalize the Hermitian $\chi(x)$ by choosing a suitable unitary matrix $U(x) \in G$

$$\chi'(x) = \mathrm{diag}(\lambda_1(x), \lambda_2(x), \cdots, \lambda_N(x)). \tag{2}$$

This is regarded as a a partial gauge fixing, if $\chi(x)$ is a gauge-dependent quantity.

(2a) At non-degenerate points $x \in \mathbb{R}^D$ of spacetime, the gauge group G is partially fixed, leaving a subgroup H unfixed, i.e., a partial gauge fixing:

$$G = SU(N) \to H = U(1)^{N-1} \times \mathrm{Weyl}. \tag{3}$$

(2b) At degenerate points $x_0 \in \mathbb{R}^D$, $\lambda_j(x_0) = \lambda_k(x_0)$ $(j \neq k = 1, \cdots, N)$, a magnetic monopole appears in the diagonal part of $\mathscr{A}_\mu(x)$ (defects of gauge fixing procedure).

$G = SU(N)$ non-Abelian Yang-Mills field
$\to H = U(1)^{N-1}$ Abelian gauge field + magnetic monopoles + electrically charged matter field ['t Hooft, 1981] e.g., $\chi(x) = \mathscr{F}_{12}(x)$, $\mathscr{F}_{\mu\nu}^2$, $\mathscr{F}_{\mu\nu}(x) D^2 \mathscr{F}_{\mu\nu}(x)$

Proof) G=SU(2).
(a) For non-degenerate points, the residual symmetry $U(1)$ exists: $U(x) = \text{diag.}(e^{i\theta_1(x)}, e^{i\theta_2(x)})$ does not change the diagonal form. $(\det U(x) = 1 \Longleftrightarrow \sum_{j=1}^{2} \theta_j(x) = 0)$ For the Cartan decomposition

$$
\begin{aligned}
\mathscr{A}_\mu =& A_\mu^3 H_1 + W_\mu^* \tilde{E}_+ + W_\mu \tilde{E}_- = \frac{1}{2} \begin{pmatrix} A_\mu^3 & \sqrt{2} W_\mu{}^* \\ \sqrt{2} W_\mu & -A_\mu^3 \end{pmatrix}, \\
W_\mu =& \frac{1}{\sqrt{2}} (A_\mu^1 + i A_\mu^2), \quad H_1 = T^3, \quad \tilde{E}_\pm = \frac{1}{\sqrt{2}} (T^1 \pm i T^2), \quad T^A = \sigma_A / 2 \qquad (4)
\end{aligned}
$$

the gauge transformation by the Cartan subgroup $U = \exp(-ig\theta H_1)$ reads

$$
\mathscr{A}_\mu{}' = U \left(\mathscr{A}_\mu + i g^{-1} \partial_\mu \right) U^\dagger = (A_\mu^3 + \partial_\mu \theta) H_1 + e^{-ig\theta} W_\mu^* \tilde{E}_+ + e^{ig\theta} W_\mu \tilde{E}_- \qquad (5)
$$

This implies the gauge transformation law

$$
A_\mu^{3\prime} = A_\mu^3 + \partial_\mu \theta, \quad W_\mu{}' = e^{ig\theta} W_\mu, \quad W_\mu^{*\prime} = e^{-ig\theta} W_\mu^*, \qquad (6)
$$

(b) For a given Hermitian and traceless matrix: $\chi(x) := \chi_A(x)\sigma_A/2$,

eigenvalues $\lambda_1(x) = -\sqrt{\chi_A(x)\chi_A(x)}/2$, $\lambda_2(x) = +\sqrt{\chi_A(x)\chi_A(x)}/2$

The degenerate point x_0, i.e., $\lambda_1(x_0) = \lambda_2(x_0)$, is determined by three equations:

$\chi_1(x_0) = \chi_2(x_0) = \chi_3(x_0) = 0 \Longleftrightarrow \chi_A(x_0) = 0 \ (A = 1, 2, 3)$

Then a magnetic monopole appears at zeros x_0 of $\chi(x)$ as follows. Around the zeros, $\chi(x)$ is expanded as

$$\chi(x) = \chi_A(x)\sigma_A/2 = (x - x_0)_j \partial_j \chi_A(x_0)\sigma_A/2 + \cdots \tag{7}$$

The matrix $U(x)$ diagonalizing $\chi(x)$ is given by choosing $\alpha = \varphi$, $\beta = \theta$, $\gamma = \gamma(\varphi)$ in the Euler angles representation:

$$U(x) = e^{-i\gamma(x)\sigma_3(x)/2} e^{-i\beta(x)\sigma_2(x)/2} e^{-i\alpha(x)\sigma_3(x)/2} = \begin{pmatrix} e^{-\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} & -e^{\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} \\ e^{-\frac{i}{2}(\alpha-\gamma)} \sin\frac{\beta}{2} & e^{\frac{i}{2}(\alpha+\gamma)} \cos\frac{\beta}{2} \end{pmatrix} \tag{8}$$

The diagonal part of

$$ig^{-1}U(x)\partial_\mu U^\dagger(x) = g^{-1}\frac{1}{2}\begin{pmatrix} \cos\beta\partial_\mu\alpha + \partial_\mu\gamma & [-i\partial_\mu\beta - \sin\beta\partial_\mu\gamma]e^{i\alpha} \\ [i\partial_\mu\beta - \sin\beta\partial_\mu\gamma]e^{i\alpha} & -[\cos\beta\partial_\mu\alpha + \partial_\mu\gamma] \end{pmatrix} = \mathscr{V}_\mu^A\sigma_A/2 \tag{9}$$

contains the singular potential of the Dirac type.

$$\mathscr{V}_\mu^3 = g^{-1}[\cos\beta\partial_\mu\alpha + \partial_\mu\gamma] \tag{10}$$

In fact, the $D = 3$ version agrees with the well-known Dirac magnetic potential:

$$\begin{aligned} ig^{-1}U(x)\nabla U^\dagger(x) =& \frac{\sigma_1}{2}\frac{g^{-1}}{r}[\sin\varphi\mathbf{e}_\theta + \cos\varphi\partial_\varphi\gamma\mathbf{e}_\varphi] \\ &+ \frac{\sigma_2}{2}\frac{g^{-1}}{r}[-\cos\varphi\mathbf{e}_\theta + \sin\varphi\partial_\varphi\gamma\mathbf{e}_\varphi] \\ &+ \frac{\sigma_3}{2}\frac{g^{-1}}{r}\frac{\cos\theta + \partial_\varphi\gamma}{\sin\theta}\mathbf{e}_\varphi \end{aligned} \tag{11}$$

D=3: 0-dimensional point defect $\rightarrow$ magnetic monopole

D=4: 1-dimensional line defect $\rightarrow$ magnetic monopole loop (closed string)

§ Maximal Abelian gauge (MAG) and magnetic monopoles

• quark confinement follows from the area law of the Wilson loop average [Wilson,1974]

$$\text{Non-Abelian Wilson loop} \quad \left\langle \mathrm{tr}\left[\mathscr{P}\exp\left\{ ig\oint_C dx^\mu \mathscr{A}_\mu(x) \right\}\right] \right\rangle_{\rm YM}^{\rm no\ GF} \sim e^{-\sigma_{NA}|S|}, \quad (1)$$

• Numerical simulation on the lattice after imposing the Maximal Abelian gauge (MAG):
for the SU(2) Cartan decomposition: $\mathscr{A}_\mu = A_\mu^a \frac{\sigma^a}{2} + A_\mu^3 \frac{\sigma^3}{2}$ $(a = 1, 2)$, $\mathscr{A}_\mu \to A_\mu^3 \frac{\sigma^3}{2}$

$$\text{Abelian-projected Wilson loop} \quad \left\langle \exp\left\{ ig\oint_C dx^\mu A_\mu^3(x) \right\} \right\rangle_{\rm YM}^{\rm MAG} \sim e^{-\sigma_{Abel}|S|} \quad !? \quad (2)$$

The continuum form of MAG is $[\partial_\mu \delta^{ab} - g\epsilon^{ab3} A_\mu^3(x)] A_\mu^b(x) = 0$ $(a, b = 1, 2)$.
· Abelian dominance $\Leftrightarrow \sigma_{Abel} \sim \sigma_{NA}$ (92±4)% [Suzuki & Yotsuyanagi,PRD42,4257,1990]

$$A_\mu^3 = \text{Monopole part} + \text{Photon part}, \quad (3)$$

· Monopole dominance $\Leftrightarrow \sigma_{monopole} \sim \sigma_{Abel}$ (95)%
[Stack, Neiman and Wensley, hep-lat/9404014][Shiba & Suzuki, hep-lat/9404015]

Maximal Abelian gauge $\equiv$ a partial gauge fixing $G = SU(N) \to H = U(1)^{N-1}$: the gauge freedom $\mathscr{A}_\mu(x) \to \mathscr{A}_\mu^\Omega(x) := \Omega(x)[\mathscr{A}_\mu(x) + ig^{-1}\partial_\mu]\Omega^{-1}(x)$ is used to transform the gauge variable as close as possible to the Abelian components for the maximal torus subgroup H of the gauge group G.

The magnetic monopole of the Dirac type appears in the diagonal part A_μ^3 of $\mathscr{A}_\mu(x)$ as defects of gauge fixing procedure.

MAG is given by minimizing the function $F_{\rm MAG}$ w.r.t. the gauge transformation Ω.

$$\min_\Omega F_{\rm MAG}[\mathscr{A}^\Omega], \quad F_{\rm MAG}[\mathscr{A}] := \frac{1}{2}(A_\mu^a, A_\mu^a) = \int d^D x \frac{1}{2} A_\mu^a(x) A_\mu^a(x) \quad (a = 1, 2) \tag{4}$$

$$\delta_\omega F_{\rm MAG} = (\delta_\omega A_\mu^a, A_\mu^a) = ((D_\mu[A]\omega)^a, A_\mu^a) = -(\omega^a, D_\mu^{ab}[A^3] A_\mu^b) \tag{5}$$

The residual U(1) exists.
cf. Lorentz gauge (Landau gauge) $G = SU(N) \to H = \{0\}$:

$$\min_\Omega F_{\rm L}[\mathscr{A}^\Omega], \quad F_L[\mathscr{A}] := \frac{1}{2}(\mathscr{A}_\mu^A, \mathscr{A}_\mu^A) = \int d^D x \frac{1}{2} \mathscr{A}_\mu^A(x) \mathscr{A}_\mu^A(x) \quad (A = 1, 2, 3) \tag{6}$$

$$\delta_\omega F_L = (\delta_\omega \mathscr{A}_\mu^A, \mathscr{A}_\mu^A) = ((D_\mu[\mathscr{A}]\omega)^A, \mathscr{A}_\mu^A) = -(\omega^A, (D_\mu[\mathscr{A}]\mathscr{A}_\mu)^A) = -(\omega^A, \partial_\mu \mathscr{A}_\mu^A)$$

$$\delta_\omega^2 F_L = -(\omega^A, \partial_\mu \delta_\omega \mathscr{A}_\mu^A) = (\omega^A, (-\partial_\mu D_\mu[\mathscr{A}])^{AB} \omega^B) \quad \text{FP operator}$$

⊙ Problems:

- The naive Abelian projection and the MAG break SU(2) color symmetry explicitly.
- Abelian dominance has never been observed in gauge fixings other than MAG.

The dual superconductivity might be a gauge artifact?

In order to establish the gauge-invariant dual superconductivity in Yang-Mills theory, we must answer the questions:

1. How to extract the "Abelian" part responsible for dual superconductivity from the non-Abelian gauge theory in the gauge-invariant way (without losing characteristic features of non-Abelian gauge theory, e.g., asymptotic freedom).

2. How to define the magnetic monopole to be condensed in Yang-Mills theory even in absence of any scalar field in the gauge-invariant way (cf. Georgi-Glashow model).

• Representation dependence is unsettled in the Abelian magnetic monopole scenario of confinement: The force between quarks depends on their $U(1)^{N-1}$ electric charges, rather than their N-ality. The asymptotic string tension between quarks of a given Abelian charge depends only on that Abelian charge, and not on the quadratic Casimir or the N-ality of the associated SU(N) representation.

Chapter II
Wilson loop
and
magnetic monopole

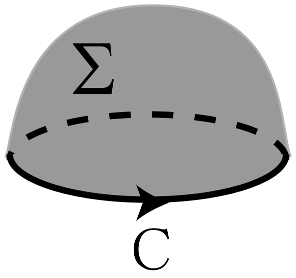
§ Wilson loop and magnetic monopole

⊙ Non-Abelian Stokes theorem for the Wilson loop

The Wilson loop operator for SU(2) Yang-Mills connection

$$W_C[\mathscr{A}] := \mathrm{tr}\left[\mathscr{P}\exp\left\{ig\oint_C dx^\mu \mathscr{A}_\mu(x)\right\}\right]/\mathrm{tr}(\mathbf{1}), \quad \mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)\sigma^A/2$$

The path-ordering $\mathscr{P}$ is removed by a non-Abelian Stokes theorem for the Wilson loop operator in the J representation of SU(2): $J = 1/2, 1, 3/2, 2, \cdots$
[Diakonov & Petrov, PLB 224, 131 (1989); hep-th/9606104]



$$W_C[\mathscr{A}] := \int d\mu_S(U)\exp\left\{iJg\int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu} f_{\mu\nu}\right\},$$ no path-ordering

$$f_{\mu\nu}(x) := \partial_\mu[\mathscr{A}_\nu^A(x)\boldsymbol{n}^A(x)] - \partial_\nu[\mathscr{A}_\mu^A(x)\boldsymbol{n}^A(x)] - g^{-1}\epsilon^{ABC}\boldsymbol{n}^A(x)\partial_\mu\boldsymbol{n}^B(x)\partial_\nu\boldsymbol{n}^C(x),$$

$$n^A(x)\sigma^A := U^\dagger(x)\sigma^3 U(x), \quad U(x) \in SU(2) \quad (A, B, C \in \{1, 2, 3\})$$

and $d\mu_S(U)$ is the product measure of an invariant measure on SU(2)/U(1) over S:

$$d\mu_S(U) := \prod_{x\in S} d\mu(U(x)), \quad d\mu(U(x)) = \frac{2J+1}{4\pi}\delta(\boldsymbol{n}^A(x)\boldsymbol{n}^A(x) - 1)d^3\boldsymbol{n}(x).$$

⊙ The geometric and topological meaning of the Wilson loop operator in D-dim. Euclidean space [K.-I.K., arXiv:0801.1274, Phys.Rev.D77:085029 (2008)] [K.-I.K., hep-th/0009152]

$$W_C[\mathscr{A}] = \int d\mu_\Sigma(U) \exp\left\{iJg(\Xi_\Sigma, k) + iJg(N_\Sigma, j)\right\}, \quad C = \partial\Sigma$$

$$k := \delta^* f = {}^*df, \quad \Xi_\Sigma := \delta^* \Theta_\Sigma \triangle^{-1} \leftarrow \quad \text{(D-3)-forms}$$

$$j := \delta f, \quad N_\Sigma := \delta \Theta_\Sigma \triangle^{-1} \leftarrow \quad \text{1-forms (D-indep.)}$$

$$\Theta_\Sigma^{\mu\nu}(x) = \int_\Sigma d^2 S^{\mu\nu}(x(\sigma))\delta^D(x - x(\sigma))$$

k and j are gauge invariant and conserved currents, $\delta k = 0 = \delta j$.

The magnetic monopole is a topological object of co-dimension 3.
D=3: 0-dimensional point defect → magnetic monopole of Wu-Yang type
D=4: 1-dimensional line defect → magnetic monopole loop (closed loop)

We do not need to use the Abelian projection ['t Hooft,1981] to define magnetic monopoles in Yang-Mills theory!

The Wilson loop operator knows the (gauge-invariant) magnetic monopole!

For $D = 3$,

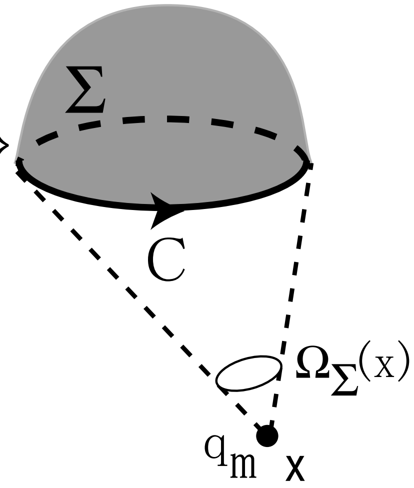
$$k(x) = \frac{1}{2}\epsilon^{jk\ell}\partial_\ell f_{jk}(x) = \rho_m(x)$$

denotes the magnetic charge density at x, and

$$\Xi_\Sigma(x) = \Omega_\Sigma(x)/(4\pi)$$

agrees with the (normalized) solid angle at the point x subtended by the surface Σ bounding the Wilson loop C. The magnetic part reads

$$W^m_{\mathscr{A}} := \exp\{iJg(\Xi_\Sigma, k)\} = \exp\left\{iJg\int d^3x\rho_m(x)\frac{\Omega_\Sigma(x)}{4\pi}\right\}$$



The magnetic charge q_m obeys the Dirac-like quantization condition :

$$q_m := \int d^3x\rho_m(x) = 4\pi g^{-1}n \quad (n \in \mathbb{Z})$$

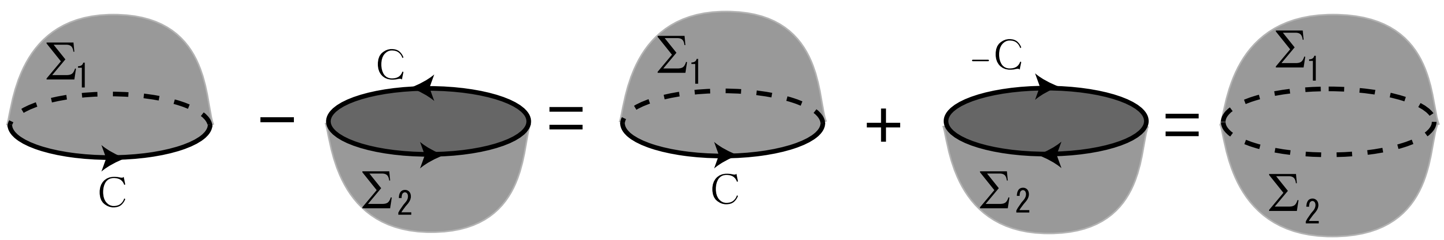
[Proof] The non-Abelian Stokes theorem does not depend on the surface Σ chosen for spanning the surface bounded by the loop C,
See [K.-I.K., arXiv0801.1274, Phys.Rev.D77:085029 (2008)]

Quantization condition for the magnetic charge [K.-I.K., arXiv0801.1274[hep-th]]

$$q_m := \int d^3x \rho_m(x) = 4\pi g^{-1} n \quad (n \in \mathbb{Z}) \tag{1}$$

the non-Abelian Stokes them does not depend on the surface Σ chosen for spanning the surface bounded by the loop C,

$$\begin{aligned} 2\pi n =& \frac{1}{2} g \int d^3x \rho_m(x) \frac{\Omega_{\Sigma_1}(x)}{4\pi} - \frac{1}{2} g \int d^3x \rho_m(x) \frac{\Omega_{\Sigma_2}(x)}{4\pi} \\ =& \frac{1}{2} g \int d^3x \rho_m(x) \frac{\Omega_{\Sigma_1}(x) - \Omega_{\Sigma_2}(x)}{4\pi} \\ =& \frac{1}{2} g \int d^3x \rho_m(x) = \frac{1}{2} g q_m, \end{aligned} \tag{2}$$



For the ensemble of point-like magnetic charges:

$$k(x) = \sum_{a=1}^{n} q_m^a \delta^{(3)}(x - z_a)$$

$$\Longrightarrow W_{\mathscr{A}}^m = \exp\left\{ iJ\frac{g}{4\pi}\sum_{a=1}^{n} q_m^a \Omega_\Sigma(z_a) \right\} = \exp\left\{ iJ\sum_{a=1}^{n} n_a \Omega_\Sigma(z_a) \right\}, \quad n_a \in \mathbb{Z}$$

The magnetic monopoles in the neighborhood of the Wilson surface Σ ($\Omega_\Sigma(z_a) = \pm 2\pi$) contribute to the Wilson loop

$$W_{\mathscr{A}}^m = \prod_{a=1}^{n} \exp(\pm i 2\pi J n_a) = \begin{cases} \prod_{a=1}^{n}(-1)^{n_a} & (J = 1/2, 3/2, \cdots) \\ = 1 & (J = 1, 2, \cdots) \end{cases}$$

$\Longrightarrow$ N-ality dependence of the asymptotic string tension
[K.-I. K., arXiv:0802.3829, J.Phys.G35:085001,2008]

Wilson loop operator is a probe of the gauge-invariant magnetic monopole defined in our formulation. Calculating the Wilson loop average reduces to the summation over the magnetic monopole charge (D=3) or current (D=4) with a geometric factor, the solid angle (D=3) or linking number (D=4).

For $D = 4$, the magnetic part reads using $\Omega_\Sigma^\mu(x)$ is the 4-dim. solid angle

$$W_{\mathscr{A}}^m = \exp\left\{ iJg \int d^4x \Omega_\Sigma^\mu(x) k^\mu(x) \right\}$$

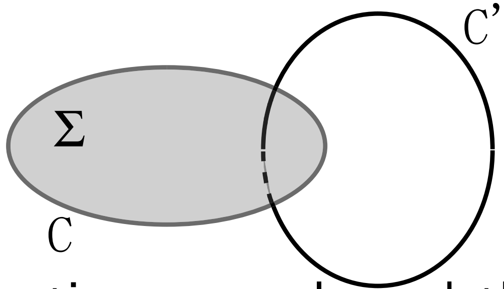
Suppose the existence of the ensemble of magnetic monopole loops C'_a,

$$k^\mu(x) = \sum_{a=1}^n q_m^a \oint_{C'_a} dy_a^\mu \delta^{(4)}(x - x_a), \quad q_m^a = 4\pi g^{-1} n_a$$

$$\Longrightarrow W_{\mathscr{A}}^m = \exp\left\{ iJg \sum_{a=1}^n q_m^a L(\Sigma, C'_a) \right\} = \exp\left\{ 4\pi Ji \sum_{a=1}^n n_a L(\Sigma, C'_a) \right\}, \quad n_a \in \mathbb{Z}$$

where $L(\Sigma, C')$ is the linking number between the surface Σ and the curve C'.

$$L(\Sigma, C') := \oint_{C'} dy^\mu(\tau) \Xi_\Sigma^\mu(y(\tau))$$



where the curve C' is identified with the trajectory of a magnetic monopole and the surface Σ with the world sheet of a hadron (meson) string for a quark-antiquark pair.

Chapter III
Reformulating Yang-Mills theory based on change of variables

§ Gauge-independent "Abelian dominance" in the Wilson loop [K.-I.K. and Shibata, arXiv0801.4203[hep-th]]

We define the *gauge-independent "Abelian dominance" in the Wilson loop*. The original gauge field is decomposed

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x),$$

so that $\mathscr{X}_\mu(x)$ does not contribute to the Wilson loop operator at all, i.e.,

$$W_C[\mathscr{A}] = \text{const.} W_C[\mathscr{V}], \tag{1}$$

where $\mathscr{V}_\mu(x)$ transforms just the same way as $\mathscr{A}_\mu(x)$ under the gauge transformation.

$W_C[\mathscr{V}]$ is written in terms of SU(2) invariant $f_{\mu\nu}(x)$ where $\mathscr{F}_{\mu\nu}[\mathscr{V}](x) = f_{\mu\nu}(x)\boldsymbol{n}(x)$:

$$W_C[\mathscr{V}] = W_S[f] := \int d\mu_S(U) \exp\left\{ iJg \int_S dS_{\mu\nu} \mathscr{F}_{\mu\nu}[\mathscr{V}](x) \cdot \boldsymbol{n}(x) \right\}. \tag{2}$$

The color field $\boldsymbol{n}(x)$ denotes a spacetime-dependent embedding of the Abelian direction into the non-Abelian color space and hence the Abelian direction can vary from point to point of spacetime.

A necessary and sufficient condition for the gauge-independent Abelian dominance for the Wilson loop operator is given by a set of conditions:

$$\text{(a)} \quad 0 = D_\mu^{[\mathscr{V}]}\boldsymbol{n}(x) := \partial_\mu \boldsymbol{n}(x) - ig[\mathscr{V}_\mu(x), \boldsymbol{n}(x)] \tag{3}$$

$$\text{(b)} \quad 0 = \mathrm{tr}\{\mathscr{X}_\mu(x)\boldsymbol{n}(x)\} \tag{4}$$

In the continuum, the gauge field is decomposed such that the Abelian dominance is given as an exact operator relation, leading to the exact (100%) Abelian dominance.

The conventional Abelian projection is reproduced for the uniform color field

$$\boldsymbol{n}(x) = \sigma_3/2, \text{ or } \mathbf{n}^A(x) = (0,0,1), \tag{5}$$

(b) implies that $\mathscr{X}_\mu(x)$ is the off-diagonal matrix: $\mathscr{X}_\mu(x) = \mathscr{A}_\mu^a(x)\frac{\sigma_a}{2}$ $(a = 1, 2)$
(a) implies that $\mathscr{V}_\mu(x)$ is the diagonal matrix: $\mathscr{V}_\mu(x) = \mathscr{A}_\mu^3\frac{\sigma_3}{2}$.
Thus, $\mathscr{A}_\mu = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x)$ reduces just to the Cartan decomposition. $f = dA^3$.

$$W_C[\mathscr{A}] = \text{const.}W_S[f] \cong \text{const.}W_C[A^3]. \tag{6}$$

Therefore, the color field $\boldsymbol{n}(x)$ plays the role of recovering color symmetry which will be lost by a global (i.e., space-time independent or uniform) choice of the Abelian direction taken in the conventional approach, e.g., the MA gauge.

§ Reformulating Yang-Mills theory in terms of new variables

SU(2) Yang-Mills theory written in terms of $\mathbb{A}_\mu^A(x)$ $(A = 1, 2, 3)$ $\Longleftrightarrow$ change of variables A reformulated Yang-Mills theory written in terms of new variables: $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ $(A = 1, 2, 3)$

We introduce a "color field" $\mathbf{n}(x)$ of unit length with three components

$$\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x)), \quad \mathbf{n}(x) \cdot \mathbf{n}(x) = n_A(x) n_A(x) = 1$$

New variables $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ should be given as functionals of the original $\mathbb{A}_\mu^A(x)$. Off-shell Cho-Faddeev-Niemi-Shabanov decomposition is reinterpreted as change of variables from $\mathbb{A}_\mu^A(x)$ to $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ via the reduction of a gauge symmetry.

Expected role of the color field:

- The color field $\boldsymbol{n}(x)$ plays the role of recovering color symmetry which will be lost in the conventional approach, e.g., the MA gauge.

- The color field $\boldsymbol{n}(x)$ carries topological defects responsible for non-perturbative phenomena, e.g., quark confinement.

Suppose that $\mathbf{n}(x)$ is given as a functional of $\mathbb{A}_\mu(x)$, i.e., $\mathbf{n}(x) = \mathbf{n}_{\mathscr{A}}(x)$. (discussed later).

By solving two defining equations among $\mathbb{V}_\mu$, $\mathbb{X}_\mu$ and $\mathbf{n}$:

(i) covariant constantness (integrability) of color field $\mathbf{n}$ in $\mathbb{V}_\mu$: $\quad D_\mu[\mathbb{V}]\mathbf{n}(x) = 0$ (1)

(ii) orthogonality of $\mathbb{X}_\mu(x)$ to $\mathbf{n}(x)$: $\quad \mathbb{X}_\mu(x) \cdot \mathbf{n}(x) = 0$ (2)

new variables are obtained

$$\mathbb{A}_\mu(x) = \mathbb{V}_\mu(x) + \mathbb{X}_\mu(x), \tag{3}$$

$$\mathbb{V}_\mu(x) = c_\mu(x)\mathbf{n}(x) + g^{-1}\partial_\mu \mathbf{n}(x) \times \mathbf{n}(x) \tag{4}$$

$$c_\mu(x) = \mathbb{A}_\mu(x) \cdot \mathbf{n}(x), \tag{5}$$

$$\mathbb{X}_\mu(x) = g^{-1}\mathbf{n}(x) \times D_\mu[\mathbb{A}]\mathbf{n}(x) \quad (D_\mu[\mathbb{A}] := \partial_\mu + g\mathbb{A}_\mu \times) \tag{6}$$

This identification was once known as the Cho-Faddeev-Niemi-Shabanov (CFNS) "decomposition". Can $\boldsymbol{n}^A(x), c_\mu(x), \mathbb{X}_\mu^A(x)$ are change of variables?

- Counting the degrees of freedom: D-dim. SU(2) Yang-Mills

before	$\mathscr{A}_\mu^A$: 3D				total 3D
after	n^A: $3-1=2$	X_μ^A: $3D-D=2D$	C_μ: D	constraint $\boldsymbol{\chi}=0$: -2	total 3D

The field strength is rewritten as

$$\mathbb{F}_{\mu\nu}[\mathbb{A}] := \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu + g\mathbb{A}_\mu \times \mathbb{A}_\nu = \mathbb{F}_{\mu\nu}[\mathbb{V}] + D_\mu[\mathbb{V}]\mathbb{X}_\nu - D_\nu[\mathbb{V}]\mathbb{X}_\mu + g\mathbb{X}_\mu \times \mathbb{X}_\nu \quad (7)$$

In particular, $\mathbb{F}_{\mu\nu}[\mathbb{V}]$ is found to be proportional to $\mathbf{n}$:

$$\mathbb{F}_{\mu\nu}[\mathbb{V}] := \partial_\mu \mathbb{V}_\nu - \partial_\nu \mathbb{V}_\mu + g\mathbb{V}_\mu \times \mathbb{V}_\nu = \mathbf{n}[\partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})] \quad (8)$$

with the magnitude:

$$f_{\mu\nu} := \mathbf{n} \cdot \mathbb{F}_{\mu\nu}[\mathbb{V}] = \partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}) \quad (9)$$

Then we can introduce a candidate of magnetic monopole current by

$$k_\mu(x) = \partial_\nu {}^* f_{\mu\nu}(x) = (1/2)\epsilon_{\mu\nu\rho\sigma}\partial_\nu f^{\rho\sigma}(x), \quad (10)$$

Remember this is the same form as the 'tHooft-Polyakov tensor for the magnetic monopole, provided that the color unit field is the normalized adjoint scalar field in the Georgi–Glashow model: $\mathbf{n}^A(x) \leftrightarrow \hat{\phi}^A(x) := \phi^A(x)/||\phi(x)||$.

The following issues must be fixed to identify the color unit field with the topological defect such as the magnetic monopole, responsible for confinement.

1. How $\mathbf{n}(x)$ is determined from $\mathbb{A}_\mu(x)$?

   [This was assumed so far. We must give a procedure to achieve this.]

2. How the mismatch between two set of variables is solved?

   [The new variables have two extra degrees of freedom which should be eliminated by imposing appropriate constraints.]

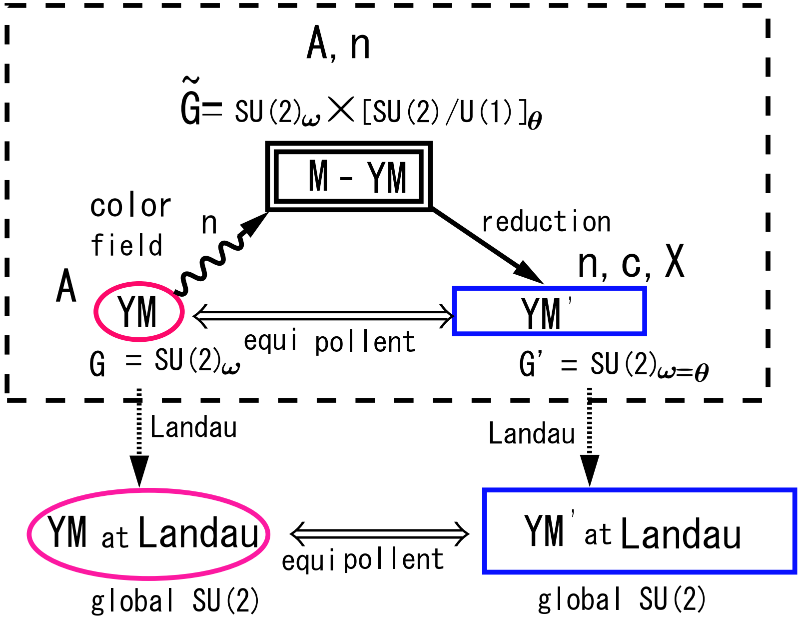
3. How the gauge transformation properties of the new variables are determined to achieve the expected one?

   [If $\mathbf{n}(x)$ transforms in the adjoint representation under the gauge transformation, $G_{\mu\nu}(x)$ becomes gauge invariant.]

All of these problems are simultaneously solved as follows.

A new viewpoint of the Yang–Mills theory

$$\delta_\theta \mathbf{n}(x) = g\mathbf{n}(x) \times \theta(x) = g\mathbf{n}(x) \times \theta_\perp(x)$$



$$\delta_\omega \mathbb{A}_\mu(x) = D_\mu[\mathbb{A}]\omega(x)$$

By introducing a color field, the original Yang-Mills (YM) theory is enlarged to the master Yang–Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G}$. By imposing the reduction condition, it is reduced to the equipollent Yang-Mills theory (YM') with the gauge symmetry G'. The overall gauge fixing condition can be imposed without breaking color symmetry, e.g. Landau gauge.

[K.-I.K., Murakami & Shinohara, hep-th/0504107; Prog.Theor.Phys. **115**, 201 (2006).]
[K.-I.K., Murakami & Shinohara, hep-th/0504198; Eur.Phys.C**42**, 475 (2005)](BRST)

As a reduction condition, we propose minimizing the functional $\int d^D x \frac{1}{2} g^2 \mathbb{X}_\mu^2$ w.r.t. enlarged gauge transformations:

$$\min_{\omega,\theta} \int d^D x \frac{1}{2} g^2 \mathbb{X}_\mu^2 = \min_{\omega,\theta} \int d^D x (D_\mu[\mathbb{A}]\boldsymbol{n})^2. \tag{1}$$

Then the infinitesimal variation reads

$$0 = \delta_{\omega,\theta} \int d^D x \frac{1}{2} \mathbb{X}_\mu^2 = -\int d^D x (\boldsymbol{\omega}_\perp - \boldsymbol{\theta}_\perp) \cdot D_\mu[\mathbb{V}]\mathbb{X}_\mu. \tag{2}$$

For $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$, the minimizing condition yields the differential form:

$$\boldsymbol{\chi} := D_\mu[\mathbb{V}]\mathbb{X}_\mu \equiv 0. \tag{3}$$

This denotes two conditions, since $\boldsymbol{n}(x) \cdot \boldsymbol{\chi}(x) = 0$ (following from $\boldsymbol{n}(x) \cdot \mathbb{X}_\mu(x) = 0$). For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$, the minimizing condition imposes no constraint.

Therefore, if we impose the reduction condition to the master-Yang–Mills theory, $\tilde{G} := SU(2)_\omega \times [SU(2)/U(1)]_\theta$ is broken down to the (diagonal) subgroup: $G' = SU(2)'$.

We have the equipollent Yang–Mills theory with the local gauge symmetry $G' := SU(2)^{\omega'}_{local}$ with $\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_{\parallel}(x), \boldsymbol{\omega}_{\perp}(x) = \boldsymbol{\theta}_{\perp}(x))$.

$$G = SU(2)^{\omega}_{local} \uparrow \tilde{G} := SU(2)^{\omega}_{local} \times [SU(2)/U(1)]^{\theta}_{local} \downarrow G' := SU(2)^{\omega'}_{local} \tag{4}$$

The reduction condition has another expression in the differential form:

$$gD_\mu[\mathbb{V}]\mathbb{X}_\mu = gD_\mu[\mathbb{A}]\mathbb{X}_\mu = D_\mu[\mathbb{A}]\{\mathbf{n} \times (D_\mu[\mathbb{A}]\mathbf{n})\} = \mathbf{n} \times (D_\mu[\mathbb{A}]D_\mu[\mathbb{A}]\mathbf{n}) = 0 \tag{5}$$

Thus, $\mathbf{n}(x)$ is determined by solving this equation for a given $\mathbb{A}_\mu(x)$. This determines the color field $\boldsymbol{n}(x)$ as a functional of a given configuration of $\mathbb{A}_\mu(x)$.

• Comparison between MAG and reduction condition:

Old MAG leaves local $U(1)_{local}(\subset G = SU(2)_{local})$ and global U(1)$_{global}$ unbroken, but breaks global SU(2)$_{global}$.

The reduction condition leaves local G'=SU(2)$_{local}$ and global SU(2)$_{global}$ unbroken (color rotation invariant)

⊙ Gauge transformation of new variables:

$$\begin{aligned}
\delta_{\omega'}\mathbf{n} =& g\mathbf{n}\times\boldsymbol{\omega'}, && \text{(6a)}\\
\delta_{\omega'}c_\mu =& \mathbf{n}\cdot\partial_\mu\boldsymbol{\omega'}, && \text{(6b)}\\
\delta_{\omega'}\mathbb{X}_\mu =& g\mathbb{X}_\mu\times\boldsymbol{\omega'}, && \text{(6c)}\\
&\Longrightarrow \delta_{\omega'}\mathbb{V}_\mu = D_\mu[\mathbb{V}]\boldsymbol{\omega'} \Longrightarrow \delta_{\omega'}\mathbb{A}_\mu = D_\mu[\mathbb{A}]\boldsymbol{\omega'}, && \text{(6d)}\\
\Longrightarrow \delta_{\omega'}\mathbb{F}_{\mu\nu}[\mathbb{V}] =& g\mathbb{F}_{\mu\nu}[\mathbb{V}]\times\boldsymbol{\omega}', && \text{(6e)}
\end{aligned}$$

Hence, the inner product $f_{\mu\nu} = \mathbf{n}\cdot\mathbb{F}_{\mu\nu}[\mathbb{V}]$ is SU(2)$'$ invariant.

$$\delta_{\omega'}f_{\mu\nu} = 0, \quad f_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu - g^{-1}\mathbf{n}\cdot(\partial_\mu\mathbf{n}\times\partial_\nu\mathbf{n}), \quad c_\mu = \mathbf{n}\cdot\mathbb{A}_\mu. \tag{7}$$

and $f_{\mu\nu}^2 = \mathbb{F}_{\mu\nu}[\mathbb{V}]^2$ is SU(2)$'$ invariant: SU(2) invariant "Abelian" gauge theory!

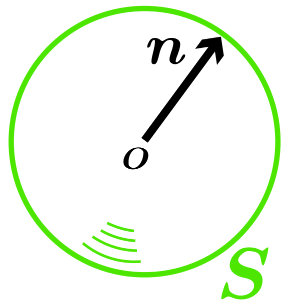
$$\delta_{\omega'}\mathbb{F}_{\mu\nu}[\mathbb{V}]^2 = \delta_{\omega'}f_{\mu\nu}^2 = 0. \tag{8}$$

Therefore, we can define the *gauge-invariant* monopole current by $k^\mu(x) := \partial_\nu{}^*f^{\mu\nu}(x) = (1/2)\epsilon^{\mu\nu\rho\sigma}\partial_\nu f_{\rho\sigma}(x)$, Moreover,

$$\delta_{\omega'}\mathbb{X}_\mu^2 = 0. \tag{9}$$

• Magnetic charge quantization:

$$\boldsymbol{n}(x) := \begin{pmatrix} n^1(x) \\ n^2(x) \\ n^3(x) \end{pmatrix} := \begin{pmatrix} \sin\beta(x)\cos\alpha(x) \\ \sin\beta(x)\sin\alpha(x) \\ \cos\beta(x) \end{pmatrix}, \tag{10}$$



The non-vanishing magnetic charge is obtained without introducing Dirac singularities in c_μ.

$$\begin{aligned} g_m :=& \int d^3x k_0 = \int d^3x \partial_i \left(\frac{1}{2} \epsilon^{ijk} f_{jk} \right) \\ =& \oint_{S^2_{phy}} d\sigma_{jk} g^{-1} \boldsymbol{n} \cdot (\partial_j \boldsymbol{n} \times \partial_k \boldsymbol{n}) = g^{-1} \oint_{S^2_{phy}} d\sigma_{jk} \sin\beta \frac{\partial(\beta,\alpha)}{\partial(x^j,x^k)} \\ =& g^{-1} \oint_{S^2_{int}} \sin\beta d\beta d\alpha = 4\pi g^{-1} n \quad (n = 0, \pm 1, \cdots) \end{aligned} \tag{11}$$

where $\frac{\partial(\beta,\alpha)}{\partial(x^\mu,x^\nu)}$ is the Jacobian: $(x^\mu, x^\nu) \in S^2_{phy} \to (\beta, \alpha) \in S^2_{int} \simeq SU(2)/U(1)$ and S^2_{int} is a surface of a unit sphere with area 4π. Hence g_m gives a number of times S^2_{int} is wrapped by a mapping from S^2_{phys} to S^2_{int}. $[\Pi_2(SU(2)/U(1)) = \Pi_2(S^2) = \mathbb{Z}]$

Chapter IV
Lattice reformulation of Yang-Mills theory and numerical simulations

§ Lattice formulation and numerical simulations

- Reformulation of Yang-Mills theory in the continuum spacetime

  [K.K., T. Murakami and T. Shinohara, hep-th/0504107 + hep-th/0504198]

- Non-compact lattice formulation

[Kato, K.K., Murakami, Shibata, Shinohara and Ito, hep-lat/0509069]

  · generation of color field configuration → Figure

  · restoration of color symmetry (global gauge symmetry) → Figure

  · gauge-invariant definition of magnetic monopole charge

- Compact lattice formulation:

[Ito, Kato, K.K., Murakami, Shibata and Shinohara, hep-lat/0604016]

  · magnetic charge quantization subject to Dirac condition $gg_m/(4\pi) \in \mathbb{Z}$ → Table

  · magnetic monopole dominance in the string tension → Table

[Shibata, Kato, K.K., Murakami, Shinohara and Ito, arXiv:0706.2529 [hep-lat]]

$M_X = 1.2 \sim 1.3\mathrm{GeV}$ ($M_A = 0.6\mathrm{GeV}$? in the Landau gauge) → Figure

- Magnetic charge quantization:

$$K(s,\mu) := 2\pi k_\mu(s) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\nu \bar{\Theta}_{\rho\sigma}(x+\mu),$$

Table 1: Histogram of the magnetic charge (value of $K(s,\mu)$) distribution for new and old monopoles on 8^4 lattice at $\beta = 2.35$.

Charge	Number(new monnopole)	Number(old monopole)
-7.5∼-6.5	0	0
-6.5∼-5.5	299	0
-5.5∼-4.5	0	1
-4.5∼-3.5	0	19
-3.5∼-2.5	0	52
-2.5∼-1.5	0	149
-1.5∼-0.5	0	1086
-0.5∼0.5	15786	13801
0.5∼1.5	0	1035
1.5∼2.5	0	173
2.5∼3.5	0	52
3.5∼4.5	0	16
4.5∼5.5	0	0
5.5∼6.5	299	0
6.5∼7.5	0	0

• String tension: magnetic monopole dominance

$$W_m(C) = \exp\left\{2\pi i \sum_{s,\mu} k_\mu(s)\Omega_\mu(s)\right\},$$

$$\Omega_\mu(s) = \sum_{s'} \Delta_L^{-1}(s-s')\frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\partial_\alpha S^J_{\beta\gamma}(s'+\hat{\mu}), \quad \partial'_\beta S^J_{\beta\gamma}(s) = J_\gamma(s), \tag{1}$$

$$V_i(R) = -\log\left\{\langle W_i(R,T)\rangle / \langle W_i(R,T-1)\rangle\right\} = \sigma_i R - \alpha_i/R + c_i \quad (i=f,m), \tag{2}$$

Table 2: String tension and Coulomb coefficient I

β	σ_f	α_f	σ_m	α_m
2.4(8^4)	0.065(13)	0.267(33)	0.040(12)	0.030(34)
2.4(16^4**)**	0.075(9)	0.23(2)	**0.068(2)**	0.001(5)

Table 3: String tension and Coulomb coefficient II
MAG+DeGrand–Toussaint (reproduced from [Stack et al., PRD 50, 3399 (1994)]

β	σ_f	α_f	σ_{DTm}	α_{DTm}
2.4(16^4)	0.072(3)	0.28(2)	**0.068(2)**	0.01(1)

• quark-antiquark potential

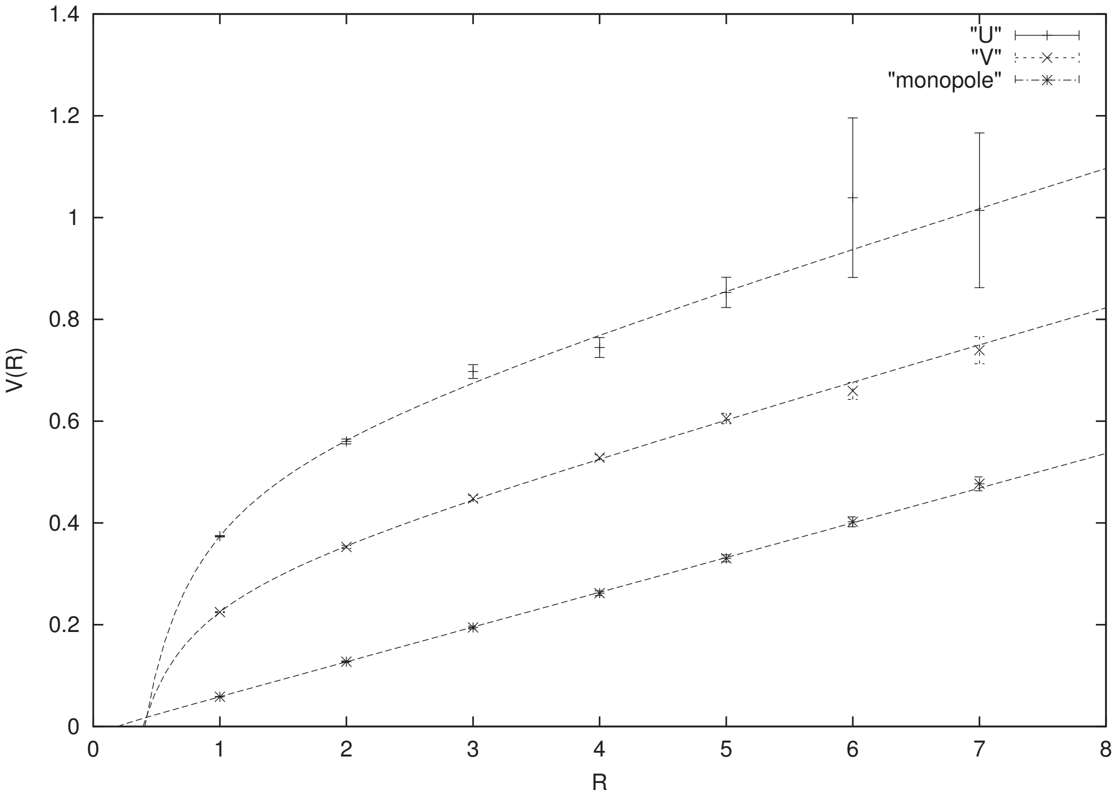


Figure 2: The full SU(2) potential $V_f(R)$, "Abelian" potential $V_a(R)$ and the magnetic–monopole potential $V_m(R)$ as functions of R at $\beta = 2.4$ on 16^4 lattice.
monopole part[Ito, Kato, K.K., Murakami, Shibata and Shinohara, hep-lat/0604016]
"Abelian" part[in preparation]

Table 4:

β	σ_f	α_f	σ_{DTm}	α_{DTm}	σ_a	α_a
2.4(16^4)	0.072(3)	0.28(2)	**0.068(2)**	0.01(1)	**0.071(3)**	0.12(1)

§ Adjoint quark potential and String breaking

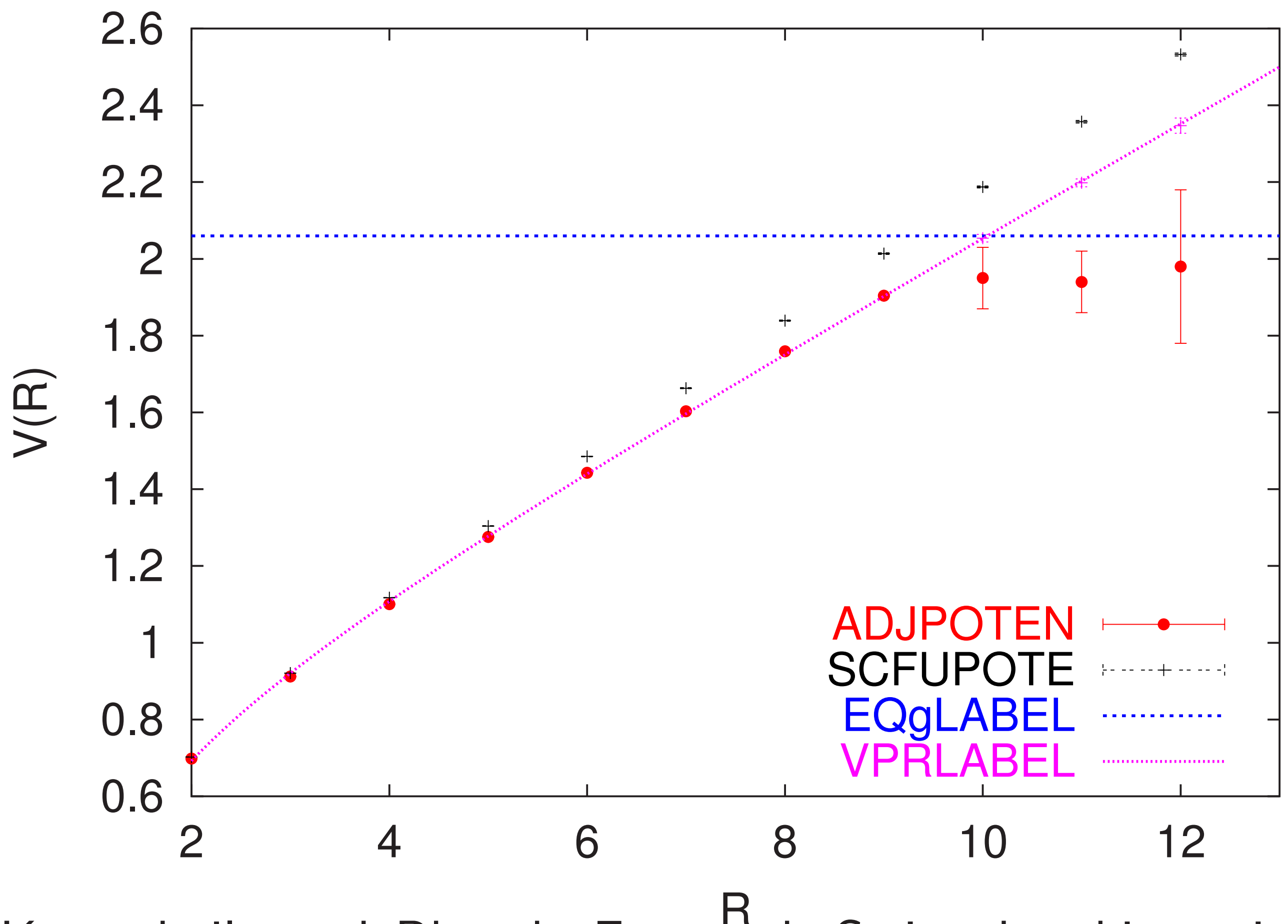


Figure 3: S. Kratochvila and Ph. de Forcrand, String breaking with Wilson loops?, hep-lat/0209094, Nucl.Phys.Proc.Suppl.119:670-672,2003
D=3, G=SU(2); The adjoint and $\frac{8}{3}$ fundamental static potentials $V(R)$ vs R. The horizontal line at 2.06(1) represents twice the energy of a gluelump.

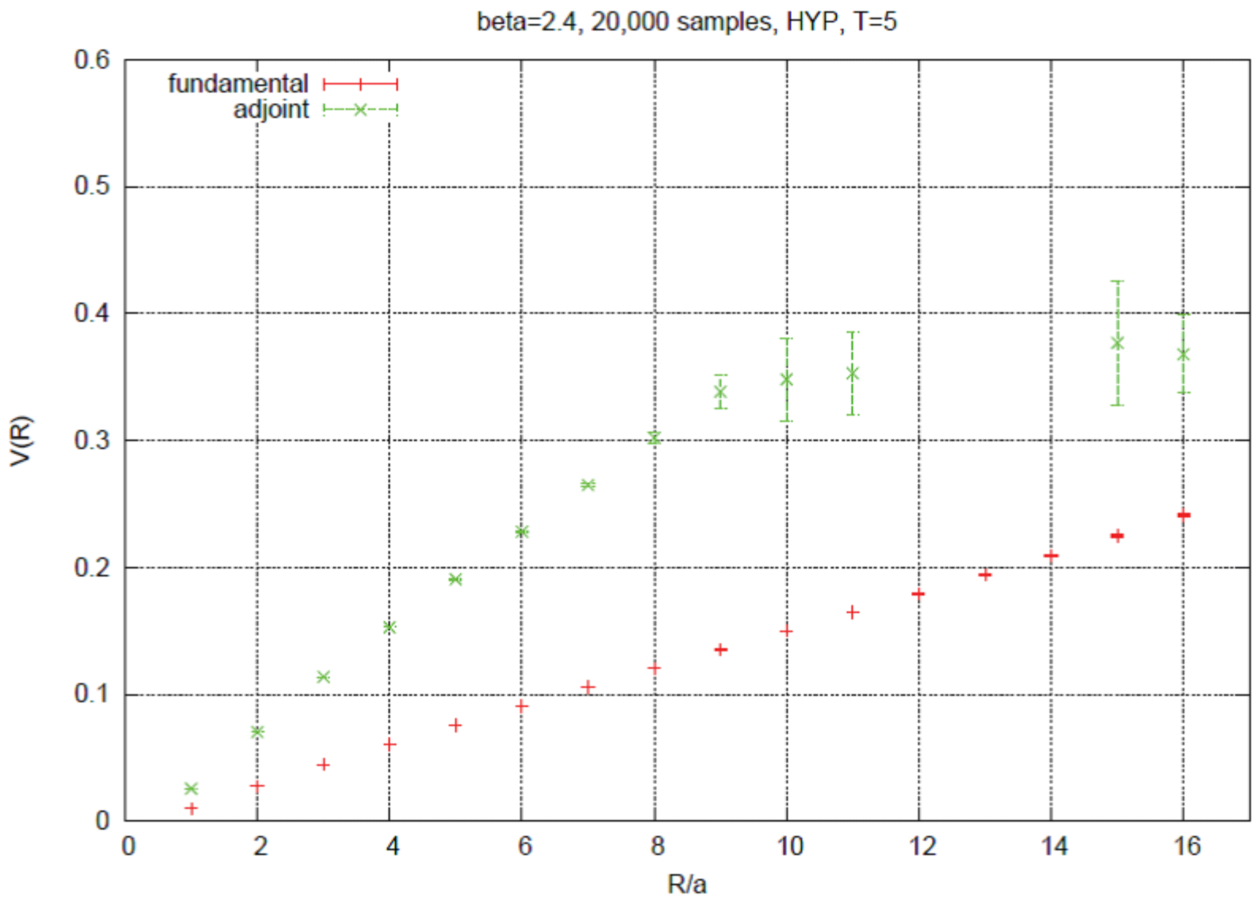


Figure 4: Our preliminary result.

- Abelian dominance in the adjoint Wilson loop? ⇒Casimir scaling, string breaking
- monopole dominance in the adjoint Wilson loop?

§ Numerical derivation of magnetic monople loops

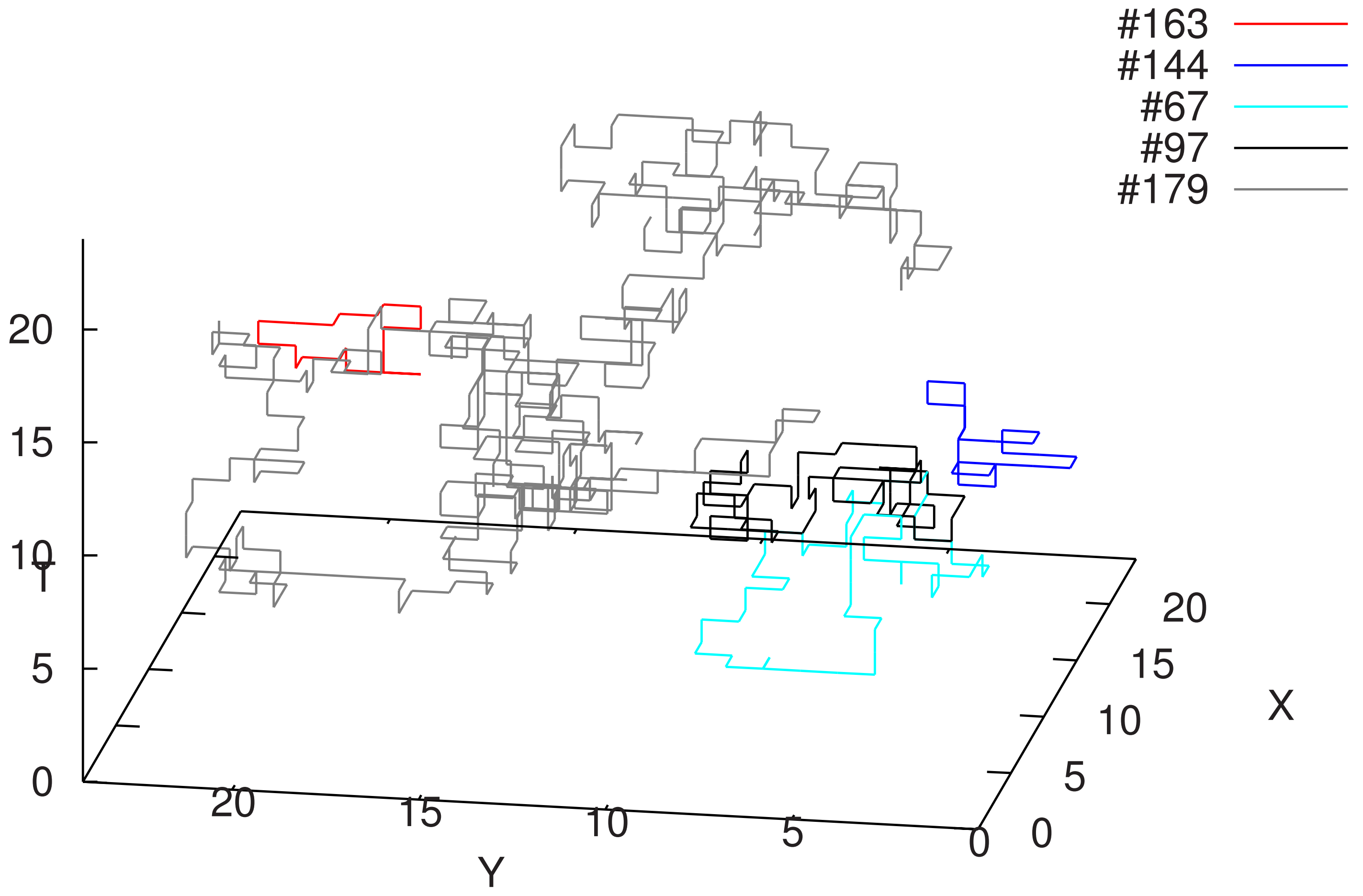


Figure 5: Preliminary result.

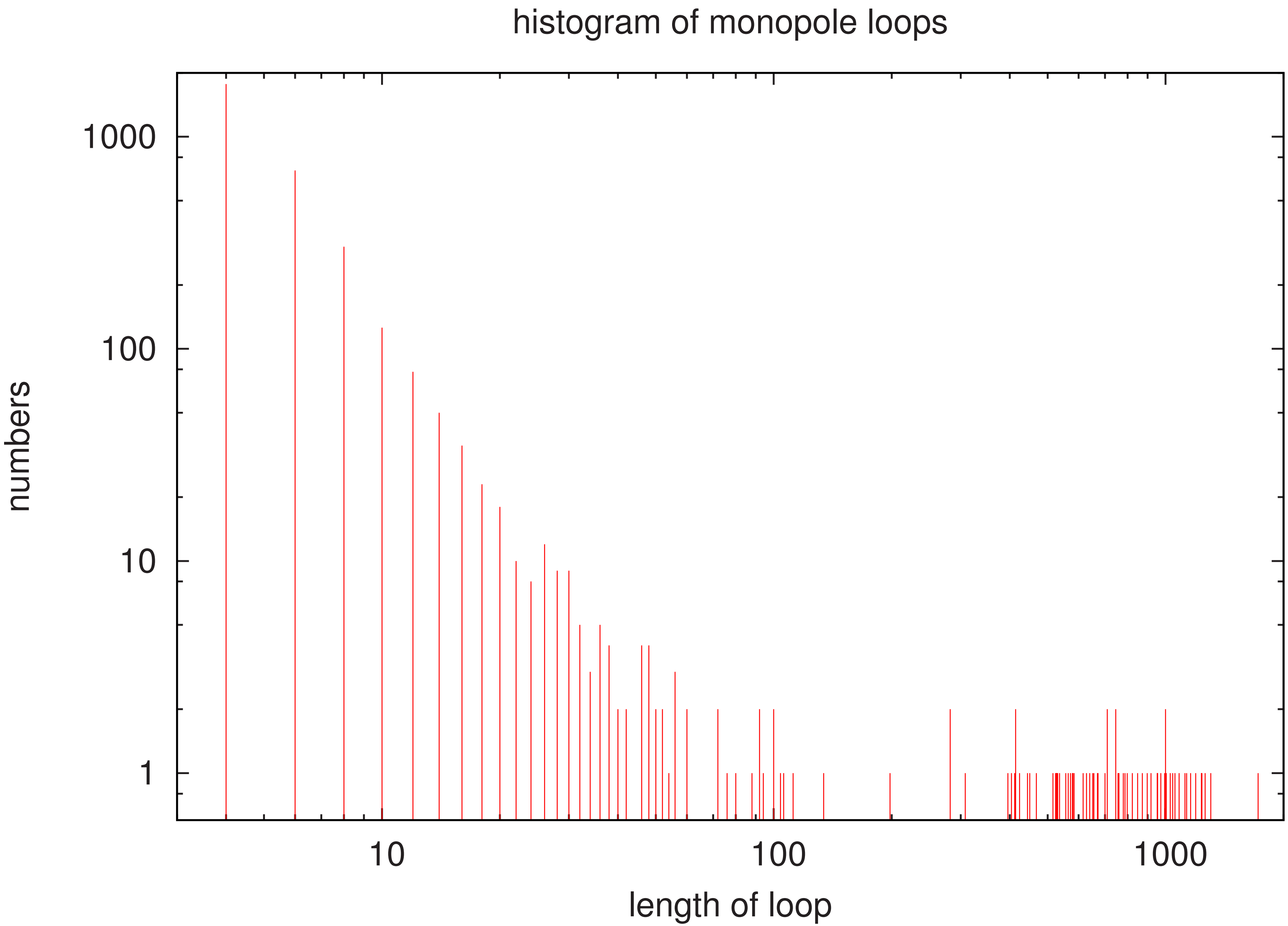
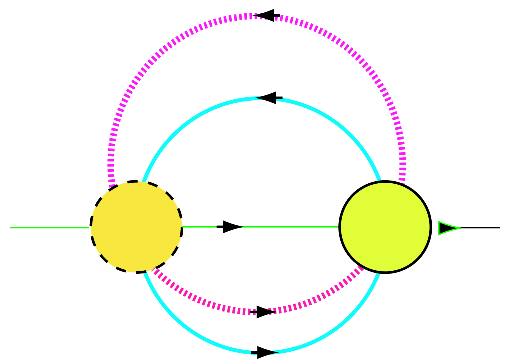


Figure 6: Preliminary result.

Chapter V
The relationship between magnetic monoole and instanton, merons, ...

§ Magnetic loops exist in the topological sector of YM_4

In the four-dimensional Euclidean SU(2) Yang-Mills theory, we give a first* (exact) analytical solution representing circular magnetic monopole loops joining two merons:



Our method reproduces also the previous results based on MAG (MCG) and LAG:
(i) the magnetic straight line can be obtained in the one-instanton or one-meorn background.
[Chernodub & Gubarev, hep-th/9506026, JETP Lett. **62**, 100 (1995).]
[Reinhardt & Tok, hep-th/0011068, Phys.Lett.B**505**, 131 (2001). hep-th/0009205.]

(ii) the magnetic closed loop can NOT be obtained in the one-instanton background.
[Brower, Orginos & Tan, hep-th/9610101, Phys.Rev.D **55**, 6313–6326 (1997)]
[Bruckmann, Heinzl, Vekua & Wipf, hep-th/0007119,Nucl.Phys.B**593**, 545–561 (2001)]

*[Bruckmann & Hansen, hep-th/0305012, Ann.Phys.**308**, 201–210 (2003)] $Q_P = \infty$

§ What are merons?

	instanton	meron
discovered by	BPST 1975	DFF 1976
$D_\nu \mathscr{F}_{\mu\nu} = 0$	YES	YES
self-duality $*\mathscr{F} = \mathscr{F}$	YES	NO
Topological charge Q_P	$(0), \pm 1, \pm 2, \cdots$	$(0), \pm 1/2, \pm 1, \cdots$
charge density D_P	$\frac{6\rho^4}{\pi^2} \frac{1}{(x^2+\rho^2)^4}$	$\frac{1}{2}\delta^4(x-a) + \frac{1}{2}\delta^4(x-b)$
solution $\mathscr{A}_\mu^A(x)$	$g^{-1}\eta^A_{\mu\nu}\frac{2(x-a)_\nu}{(x-a)^2+\rho^2}$	$g^{-1}\left[\eta^A_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2} + \eta^A_{\mu\nu}\frac{(x-b)_\nu}{(x-b)^2}\right]$
Euclidean	finite action $S_{\rm YM} = (8\pi^2/g^2)\|Q_P\|$	(logarithmic) divergent action
tunneling	between $Q_P = 0$ and $Q_P = \pm 1$ vacua in the $\mathscr{A}_0 = 0$ gauge	$Q_P = 0$ and $Q_P = \pm 1/2$ vacua in the Coulomb gauge
multi-charge solutions	Witten, 't Hooft, Jackiw-Nohl-Rebbi, ADHM	??? not known
Minkowski	trivial	everywhere regular finite, non-vanishing action

An instanton dissociates into two merons?

§ Relevant works (excluding numerical simulations)

papers	original configuration	dual counterpart	method
CG95	one instanton	a straight magnetic line	MAG (analytical)
BOT96	one instanton	no magnetic loop	MAG (numerical)
BHVW00	one instanton	no magnetic loop	LAG (analytical)
RT00	one meron	a straight magnetic line	LAG (analytical)
BOT96	instaton-antiinstanton instaton-instaton	a magnetic loop a magnetic loop	MAG (numerical) MAG (numerical)
RT00	instaton-antiinstanton	two magnetic loops	LAG (numerical)
Ours KFSS08 0806.3913 [hep-th]	one instanton one meron two merons	no magnetic loop a straight magnetic line circular magnetic loops	New (analytical) New (analytical) New (analytical)

CG95=Chernodub & Gubarev, [hep-th/9506026], JETP Lett. **62**, 100 (1995).
BOT96=Brower, Orginos & Tan, [hep-th/9610101], Phys.Rev.D **55**, 6313–6326 (1997).
BHVW00=Bruckmann, Heinzl, Vekua & Wipf, [hep-th/0007119], Nucl.Phys.B **593**, 545–561 (2001). Bruckmann, [hep-th/0011249], JHEP 08, 030 (2001).
RT00=Reinhardt & Tok, Phys.Lett. B**505**, 131–140 (2001). hep-th/0009205.
BH03=Bruckmann & Hansen, [hep-th/0305012], Ann.Phys. **308**, 201–210 (2003).

§ Derivation of RDE

Consider the enlarged gauge transformation for $\delta \boldsymbol{A}_\mu$ and $\delta \boldsymbol{\phi}$ given by

$$\delta_\omega \mathbf{A}_\mu(x) = D_\mu[\mathbf{A}]\boldsymbol{\omega}(x), \quad \delta_\theta \boldsymbol{\phi}(x) = g\boldsymbol{\phi}(x) \times \boldsymbol{\theta}_\perp(x) \quad (\boldsymbol{\omega} \in SU(2), \boldsymbol{\theta}_\perp \in SU(2)/U(1)),$$

We minimize the functional $F[\mathbf{A}, \boldsymbol{\phi}] = \int d^D x \frac{1}{2}(D_\mu[\mathbf{A}]\boldsymbol{\phi}) \cdot (D_\mu[\mathbf{A}]\boldsymbol{\phi})$, with respect to the enlarged gauge transformation as

$$0 = \delta F[\mathbf{A}, \boldsymbol{\phi}] = g \int d^D x (D_\mu[\mathbf{A}]\boldsymbol{\phi}) \cdot D_\mu[\mathbf{A}]\{\boldsymbol{\phi} \times (\boldsymbol{\theta}_\perp - \boldsymbol{\omega}_\perp)\}.$$

The integration by parts yields

$$0 = \delta F[\mathbf{A}, \boldsymbol{\phi}] = g \int d^D x (\boldsymbol{\theta}_\perp - \boldsymbol{\omega}_\perp) \cdot (\boldsymbol{\phi} \times D_\mu[\mathbf{A}] D_\mu[\mathbf{A}]\boldsymbol{\phi}) \,.$$

Therefore, this functional is invariant if $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$. Thus, if $\boldsymbol{\theta}_\perp \neq \boldsymbol{\omega}_\perp$, the minimization of the functional is achieved by $\boldsymbol{\phi}$ and $\boldsymbol{A}$ satisfying the differential equation:

$$\boldsymbol{\phi} \times D_\mu[\mathbf{A}] D_\mu[\mathbf{A}]\boldsymbol{\phi} = 0.$$

This equation gives two conditions, since it is perpendicular to $\boldsymbol{\phi}(x)$.

§ Bridge between $\mathbf{A}_\mu(x)$ and $\mathbf{n}(x)$

For a given Yang-Mills field $\mathbf{A}_\mu(x)$, the color field $\mathbf{n}(x)$ is obtained by solving the reduction differential equation (RDE):

$$\mathbf{n}(x) \times D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x) = \mathbf{0}.$$

[K.-I. K., Shinohara & Murakami, 0803.1076, Prog. Theor. Phys. 120,1–50 (2008)]

For a given SU(2) Yang-Mills field $\mathbf{A}_\mu(x) = \mathbf{A}_\mu^A(x)\frac{\sigma_A}{2}$, look for unit vector fields $\mathbf{n}(x)$ such that $-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x)$ is proportional to $\mathbf{n}(x)$: an eigenvalue-like form:

$$-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x) = \lambda(x)\mathbf{n}(x) \quad (\lambda(x) \geq 0).$$

The solution is not unique. Choose the solution giving the smallest value of the reduction functional F_{rc}: integral of the scalar function $\lambda(x)$ over the spacetime $\mathbb{R}^D$:

$$\begin{aligned} F_{\mathrm{rc}} &= \int d^D x \frac{1}{2}(D_\mu[\mathbf{A}]\mathbf{n}(x)) \cdot (D_\mu[\mathbf{A}]\mathbf{n}(x)) \\ &= \int d^D x \frac{1}{2}\mathbf{n}(x) \cdot (-D_\mu[\mathbf{A}]D_\mu[\mathbf{A}]\mathbf{n}(x)) \\ &= \int d^D x \frac{1}{2}\mathbf{n}(x) \cdot \lambda(x)\mathbf{n}(x) = \int d^D x \frac{1}{2}\lambda(x). \end{aligned}$$

We adopt the CFtHW Ansatz:

$$g\mathbf{A}_\mu(x) = \frac{\sigma_A}{2} g\mathbf{A}_\mu^A(x) = \frac{\sigma_A}{2}\eta^A_{\mu\nu} f_\nu(x), \quad f_\nu(x) := \partial_\nu \ln \Phi(x) = x_\nu f(x),$$

The new form of the RDE reduces to

$$\{-\partial_\mu\partial_\mu\delta_{AB} + 2f(x)(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)_{AB} + x^2 f^2(x)(\mathbf{S}^2)_{AB}\}\mathbf{n}_B(x) = \lambda(x)\mathbf{n}_A(x),$$
$$L_A := -\frac{i}{2}\eta^A_{\mu\nu} x_\mu \partial_\nu, \quad (S_A)_{BC} := i\epsilon_{ABC}, \quad J_A := L_A + S_A,$$

A complete set of commuting observables is given by the Casimir operators, $\vec{J}^2$, $\vec{L}^2$, $\vec{S}^2$ and their projections, e.g., J_z, L_z, S_z.

Separating $\mathbb{R}^4$ into the radial and angular parts:

$$R := \sqrt{x_\mu x_\mu} \in \mathbb{R}_+, \quad \hat{x}_\mu := x_\mu / R \in S^3,$$

$\mathbf{n}(x)$ is constructed from the vector spherical harmonics $Y^A_{(J,L)}(\hat{x})$, a polynomial in $\hat{x}$ of degree $2L$ with $(2J+1)(2L+1)$ fold generacy.

§ Merons and instantons [DeAlfaro, Fubini and Furlan, 1976, 1977]

One meron solution at the origin $x=0$ (non pure gauge everywhere)

$$\mathbf{A}_\mu^{\mathrm{M}}(x) = g^{-1}\eta_{\mu\nu}^{A}\frac{x_\nu}{x^2}\frac{\sigma_A}{2} = \frac{1}{2}ig^{-1}U(x)\partial_\mu U^{-1}(x), \quad U(x) = \frac{\bar{e}_\alpha x_\alpha}{\sqrt{x^2}} \in SU(2)$$

$$D_P(x) := \frac{1}{16\pi^2}\mathrm{tr}(\mathbf{F}_{\mu\nu} * \mathbf{F}_{\mu\nu}) = \frac{1}{2}\delta^4(x), \quad Q_p := \int d^4x D_P(x) = \frac{1}{2}.$$

$\downarrow$ Conformal transformation : $x_\mu \to z_\mu = 2a^2\dfrac{(x+a)_\mu}{(x+a)^2} - a_\mu,$

meron-antimeron solution (one meron at $x=a$ and one antimeron at $x=-a$)

$$\mathbf{A}_\mu^{\mathrm{M}}(x) \to \partial_\mu z_\nu \mathbf{A}_\nu^{\mathrm{M}}(z) = \text{complicated expressions}$$

$\downarrow$ Singular gauge transformation : $U(x+a),$

meron-meron or dimeron solution (one meron at $x=a$ and another meron at $x=-a$)

$$\mathbf{A}_\mu^{\mathrm{MM}}(x) = -g^{-1}\left[\eta_{\mu\nu}^{A}\frac{(x+a)_\nu}{(x+a)^2} + \eta_{\mu\nu}^{A}\frac{(x-a)_\nu}{(x-a)^2}\right]\frac{\sigma_A}{2}, \quad D_P(x) = \frac{1}{2}\delta^4(x+a) + \frac{1}{2}\delta^4(x-a)$$

For $a \to 0$, $\mathbf{A}_\mu^{\mathrm{MM}}(x) \to$ one-instanton in the regular gauge with zero size $g^{-1}\eta_{\mu\nu}^{A}\frac{2x_\nu}{x^2}$.

§ One-instanton and one-meron

⊙ For one-instanton in the regular gauge with zero size, the solution is 3-fold ($\hat{a}_{B,B=1,2,3}$) degenerate ($(J,L)=(0,1)$), a linear combination of the standard Hopf map:

$$\mathbf{n}_A(x) = \sum_{B=1,2,3} \hat{a}_B \sum_{\alpha,\beta,\gamma=1,2,3,4} \hat{x}_\alpha \hat{x}_\beta \bar{\eta}^B_{\alpha\gamma} \eta^A_{\gamma\beta}, \quad \lambda(x) = 0, \qquad \hat{x}_\mu := x_\mu / \sqrt{x^2}$$

The standard Hopf map is singular only at the ceter of the instanton. Therefore, k_μ is non-zero only at the ceter of the instanton and there is no magnetic monopole loop.

⊙ For one-instanton in the singular gauge with zero size, the solution is 3-fold degenerate ($(J,L)=(1,0)$)

$$\mathbf{n}_A(x) = c_A, \quad \lambda(x) = 0$$

⊙ For one-meron, the solution is 4-fold degenerate ($(J,L)=(1/2,1/2)$) 4d hedgehog:

$$\mathbf{n}_A(x) = \sum_{\nu=1,2,3,4} b_\nu \sum_{\mu=1,2,3,4} \eta^A_{\mu\nu} \hat{x}_\mu / \sqrt{b^2 - (b\cdot\hat{x})^2}, \quad \lambda_{(1/2,1/2)}(x) = \frac{2(b\cdot x)^2}{x^2[b^2x^2 - (b\cdot x)^2]}$$

k_μ denotes a straight magnetic line going through the center of the meron in the direction b_μ. The solution of RDE is not unique. The Hopf map is also a solution of RDE with $\lambda_{(0,1)}(x) = \frac{2}{x^2}$, but it is excluded, since it gives larger $F_{\mathrm{rc}} = \int d^4x \lambda(x)$.

§ (UV) Smeared meron pair [Callan, Dashen and Gross, 1978]

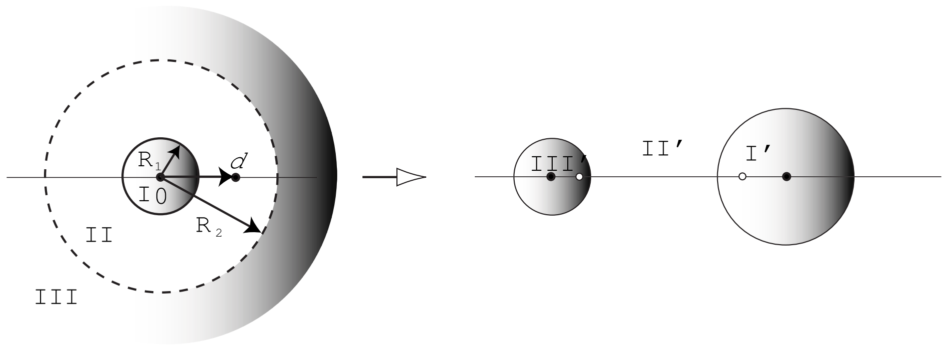


Figure 7: The concentric sphere geometry for a smeared meron (left panel) is transformed to the smeared two meron configuration (right panel) by the conformal transformation including the inversion about the point d.

$$\mathbf{A}_\mu^{\rm sMM}(x) = \frac{\sigma_A}{2}\eta^A_{\mu\nu}x_\nu \times \begin{cases} \frac{2}{x^2+R_1^2} & \text{I: } \sqrt{x^2} < R_1 \Longrightarrow Q_P^{\rm I} = \frac{1}{2} \\ \frac{1}{x^2} & \text{II: } R_1 < \sqrt{x^2} < R_2 \Longrightarrow Q_P^{\rm II} = 0\,, \\ \frac{2}{x^2+R_2^2} & \text{III: } \sqrt{x^2} > R_2 \Longrightarrow Q_P^{\rm III} = \frac{1}{2} \end{cases}$$

$$S_{\rm YM}^{\rm sMM} = \frac{8\pi^2}{g^2} + \frac{3\pi^2}{g^2}\ln\frac{R_2}{R_1},,$$

One-instanton limit: $|R_1 - R_2| \downarrow 0$ $(R_2/R_1 \downarrow 1)$. $S_{\rm YM}^{\rm sMM} = \frac{8\pi^2}{g^2}$ finite

One-meron limit: $R_2 \uparrow \infty$ or $R_1 \downarrow 0$ $(R_2/R_1 \uparrow \infty)$. $S_{\rm YM}^{\rm sMM}$ logarithmic divergence

§ Circular magnetic monopole loops joining the UV smeared meron pair

The RDE is conformal covariant and gauge covariant, while the reduction functional is conformal invariant and gauge invariant.
The minimum of the reduction functional is achieved by the solution

$$\lambda(x) = \begin{cases} \frac{8x^2}{(x^2+R_1^2)^2} & \text{I: } 0 < \sqrt{x^2} < R_1; \ \mathbf{n}_{\mathrm{I}}(x) = Y_{(1,0)} = \text{const.} \\ \frac{2(\hat{b}\cdot x)^2}{x^2[x^2-(\hat{b}\cdot x)^2]} & \text{II: } R_1 < \sqrt{x^2} < R_2; \ \mathbf{n}_{\mathrm{II}}(x) \simeq Y_{(1/2,1/2)} = \text{hedgehog .} \\ \frac{8R_2^2}{x^2(x^2+R_2^2)^2} & \text{III: } R_2 < \sqrt{x^2}; \ \mathbf{n}_{\mathrm{III}}(x) = Y_{(0,1)}(x) = \text{Hopf} \end{cases}$$

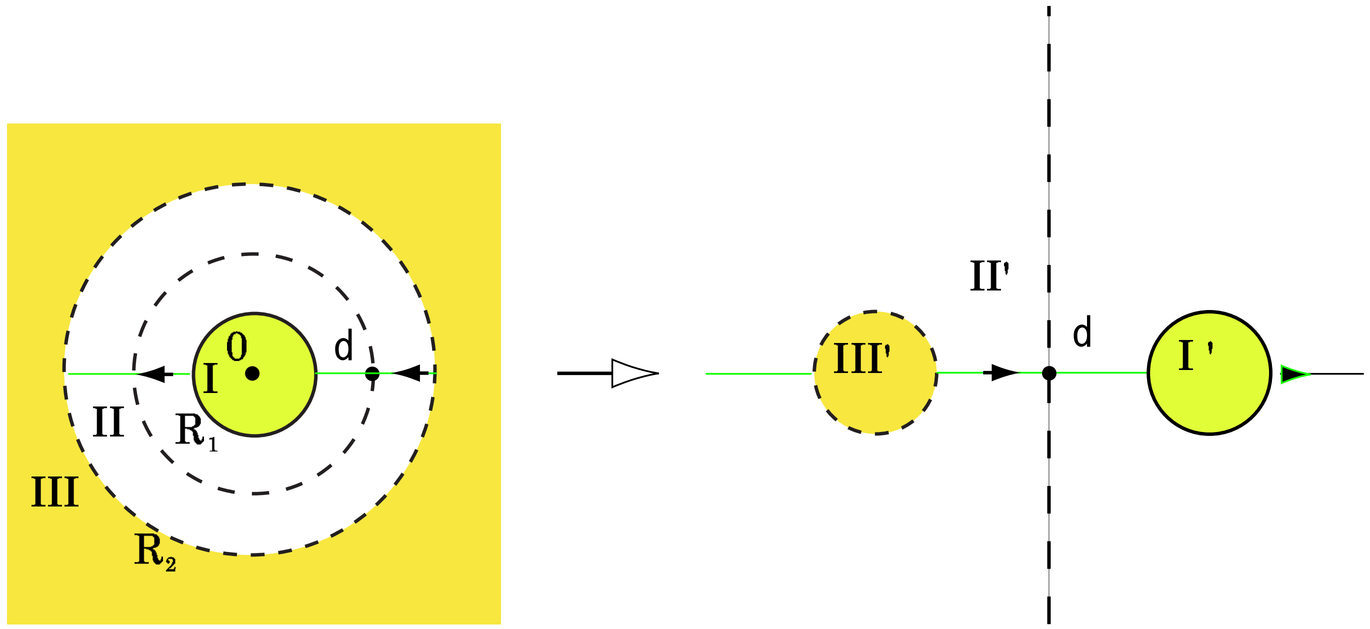
$$F_{\mathrm{rc}} = \int_{\mathbb{R}^4} d^4x \lambda(x) < \infty \quad \text{for } R_1, R_2 > 0.$$

Using the conformal transformation and the singular gauge transformation,

$$\bar{\mathbf{n}}(x)_{\mathrm{II'}} = \frac{2a^2}{(x+a)^2} \hat{b}_\nu \eta^A_{\mu\nu} z_\mu U^{-1}(x+a)\sigma_A U(x+a) / \sqrt{z^2 - (\hat{b}\cdot z)^2},$$

where

$$z_\mu = 2a^2 \frac{(x+a)_\mu}{(x+a)^2} - a_\mu, \quad U(x+a) = \frac{\bar{e}_\alpha (x+a)_\alpha}{\sqrt{(x+a)^2}},$$

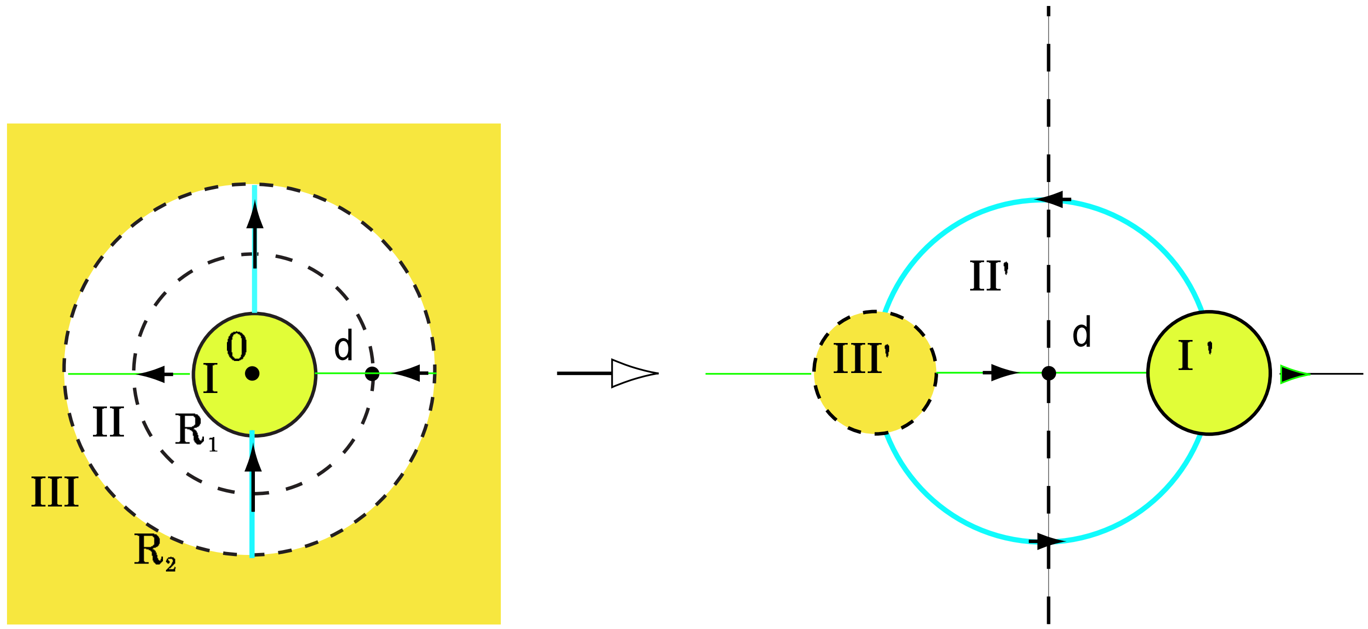


Without loss of generality, we can fix the direction of connecting two merons as $a_\mu := d_\mu/2 = \delta_{\mu 4}T$.

If $\hat{b}_\mu$ is parallel to a_μ, i.e., $\hat{b}_\mu = \delta_{\mu 4}$ (or $\hat{\mathbf{b}} = \mathbf{0}$),

$$x_A = 0 \quad (A = 1, 2, 3) \tag{1}$$

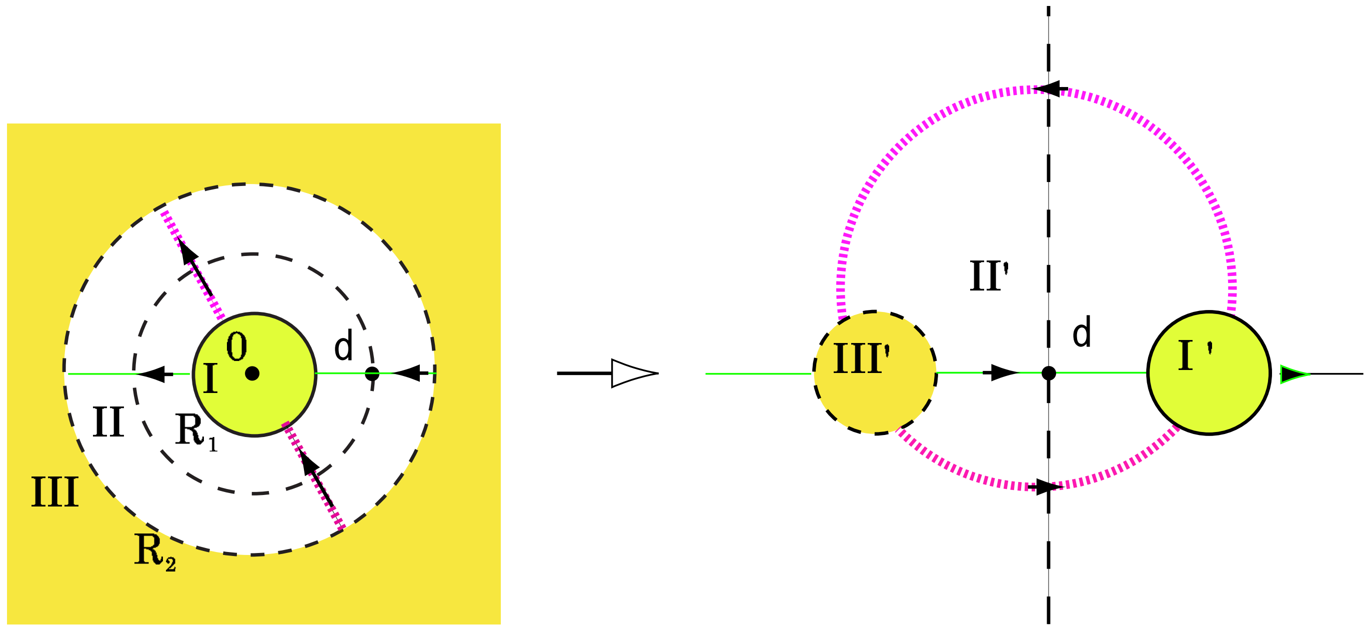
i.e., the magnetic current is a straight line going through two merons at $(\mathbf{0}, \pm T)$.



If $\hat{b}_\mu$ is perpendicular to a_μ (or $\hat{b}_\mu = \delta_{\mu\ell}\hat{b}_\ell$, $\ell = 1, 2, 3$), i.e., $\hat{b}_4 = 0$,

$$x_\ell^2 + x_4^2 = a^2. \tag{2}$$

a circular magnetic monopole loop with its center at the origin 0 in z space and the radius $\sqrt{a^2}$ joining two merons at $(\mathbf{0}, \pm T)$ exists on the plane spanned by a_μ and $\hat{b}_\ell$ $(\ell = 1, 2, 3)$.



Other chices of $\hat{b}_\mu = (\hat{\mathbf{b}}, \hat{b}_4)$

$$\mathbf{x} \times \hat{\mathbf{b}} = \mathbf{0} \quad \& \quad \left(\mathbf{x} + \frac{a \cdot \hat{b}}{|\hat{\mathbf{b}}|} \frac{\hat{\mathbf{b}}}{|\hat{\mathbf{b}}|} \right)^2 + x_4^2 = \left(a^2 + \frac{(a \cdot \hat{b})^2}{|\hat{\mathbf{b}}|^2} \right), \tag{3}$$

where $\hat{\mathbf{b}}$ is the three-dimensional part of unit four $\hat{b}_\mu$ $(\hat{b}_\mu \hat{b}_\mu = \hat{b}_4^2 + |\hat{\mathbf{b}}|^2 = 1)$.
These equations express circular magnetic monopole loops
the center at $\mathbf{x} = -\frac{a \cdot \hat{b}}{|\hat{\mathbf{b}}|} \frac{\hat{\mathbf{b}}}{|\hat{\mathbf{b}}|}$, $x_4 = 0$ with the radius $\sqrt{a^2 + \frac{(a \cdot \hat{b})^2}{|\hat{\mathbf{b}}|^2}} (\geq \sqrt{a^2})$
joining two merons at $\pm a_\mu$ on the plane specified by a_μ and $\hat{\mathbf{b}}$

§ Numerical derivation of magnetic monople loops

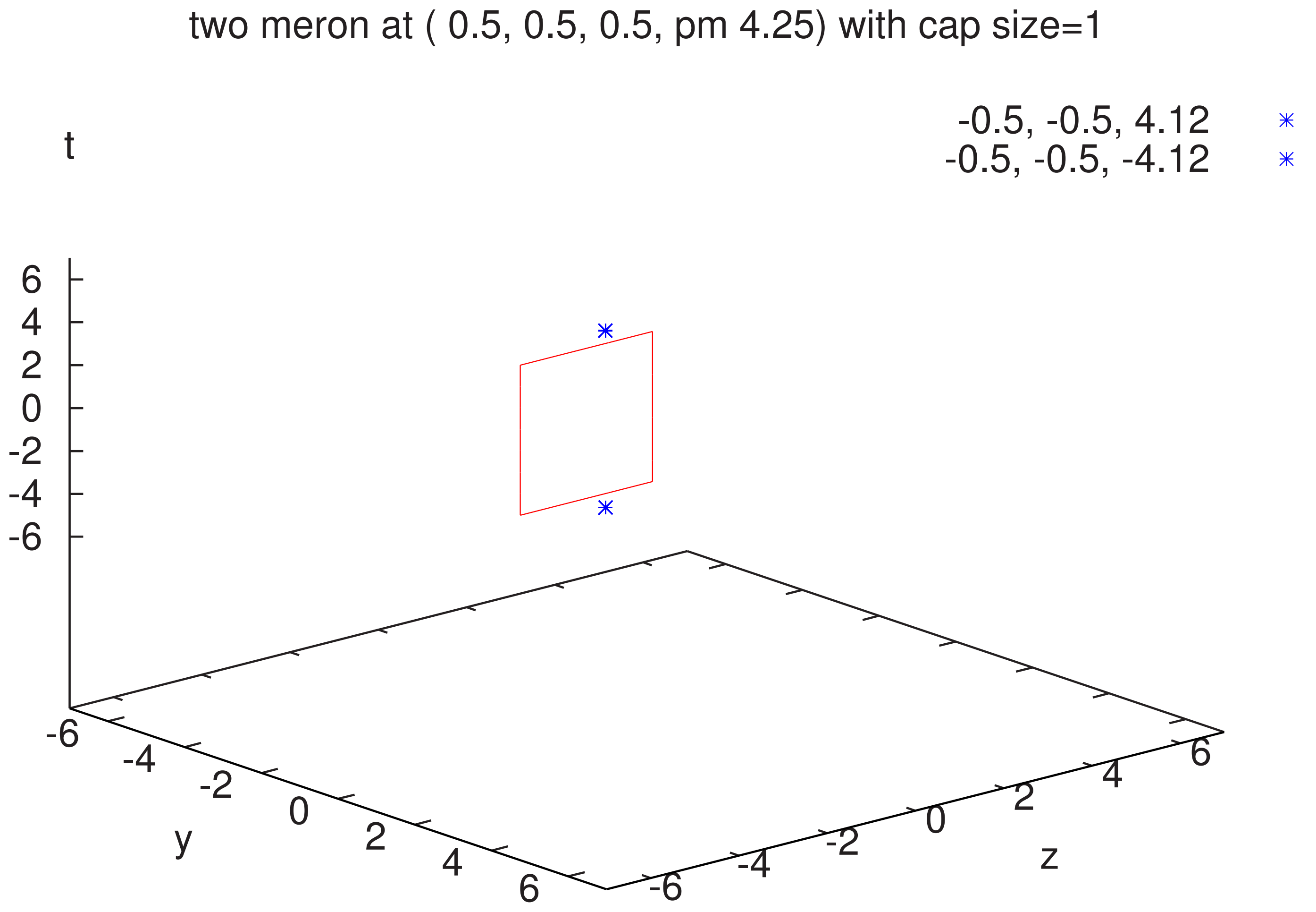


Figure 8: Numerical derivation of magnetic monople loops in the topological background, Preliminary result (in the last week)

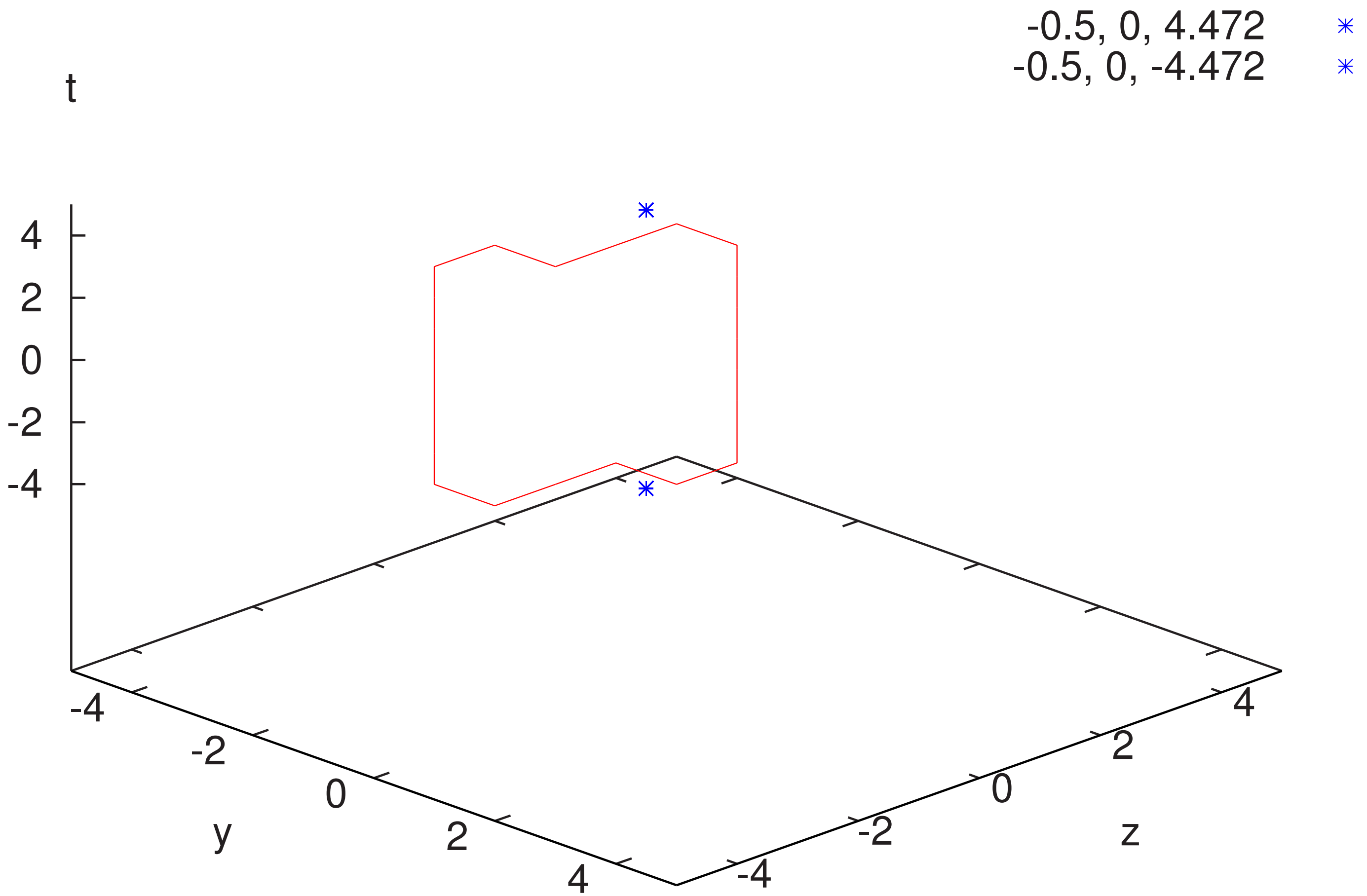


Figure 9:

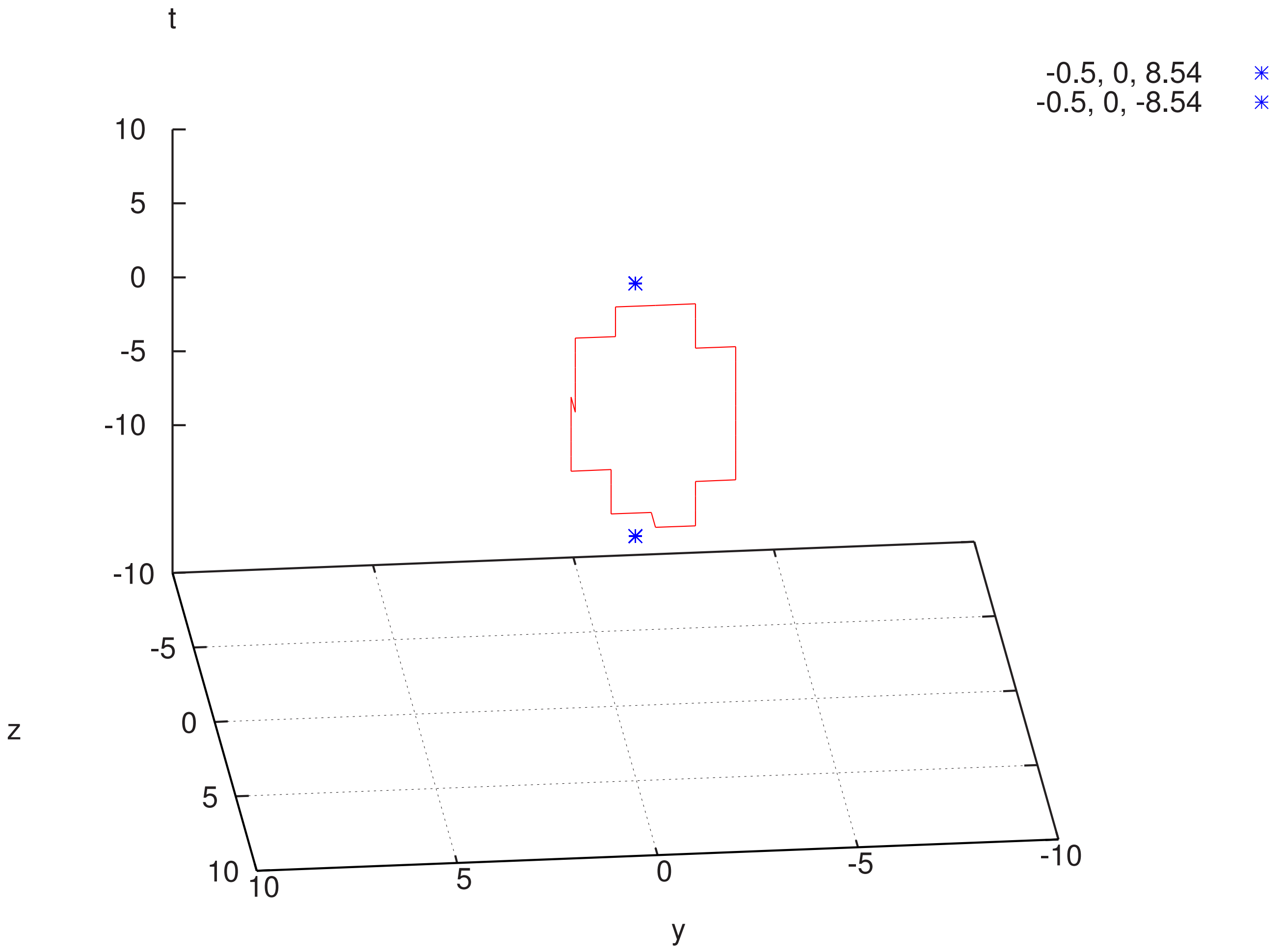


Figure 10:

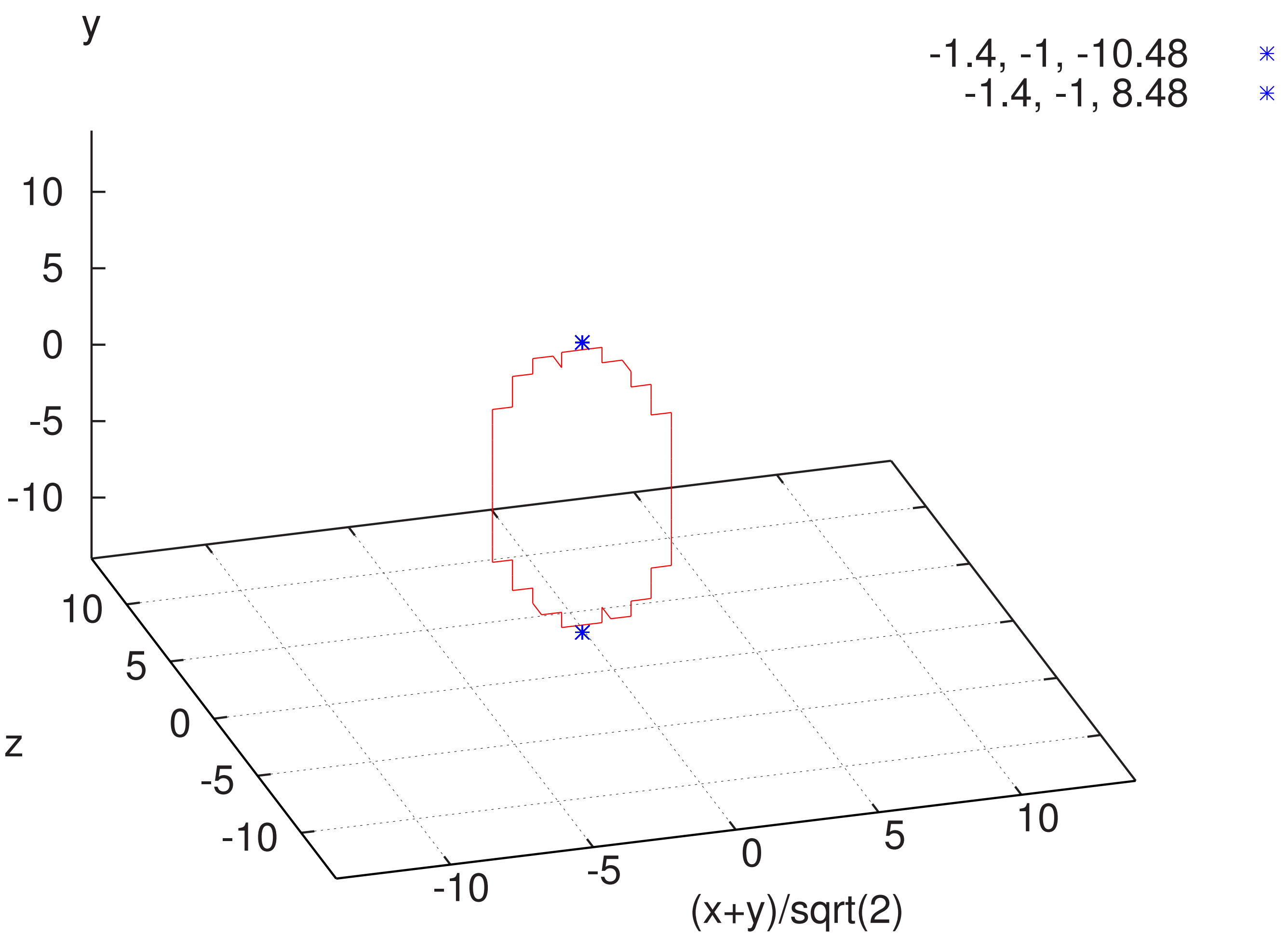


Figure 11: [-14:14][-14:14][-14:14][-24:24] 32^3 * 52 two merons at(-0.0, -0.5,-0.5 ,8.987), (-0.0, -0.5,-0.5 ,-.9.978), cap size = 3

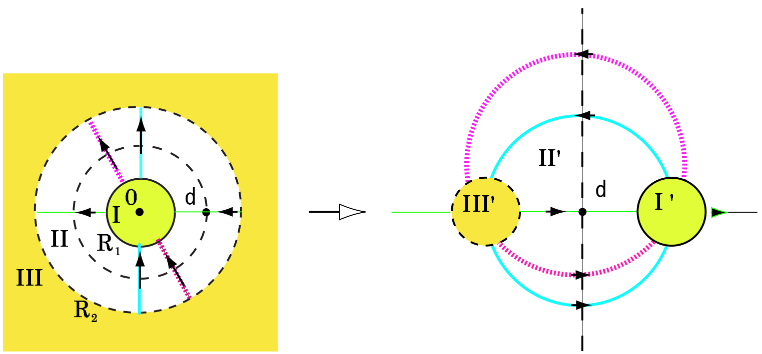
§ Conclusion and discussion

We have developed a reformulation of the Yang-Mills theory based on change of variables (a la Cho-Faddeev-Niemi) and a non-Abelian Stokes theorem (of the Diakonov-Petrov type).

In the four-dimensional Euclidean SU(2) Yang-Mills theory,

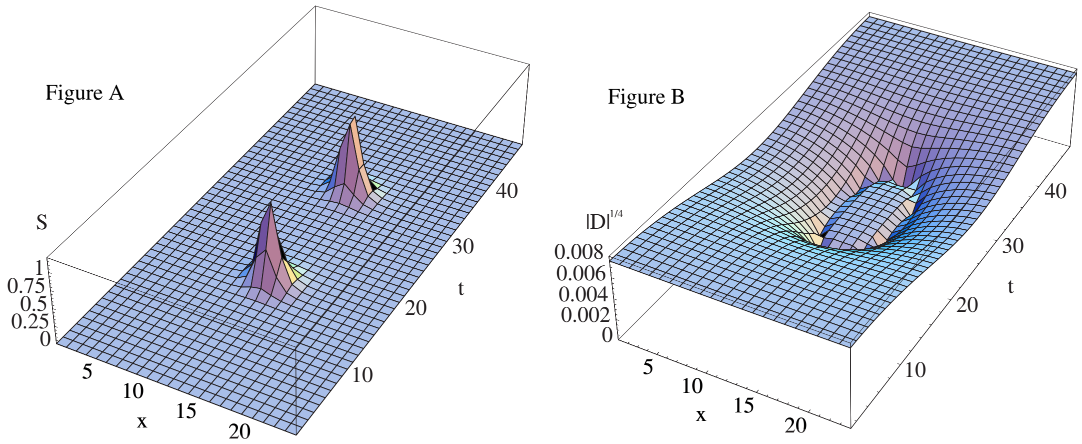
(1) we have confirmed a gauge-independent "Abelian" Dominance and magnetic monopole dominance in the Wilson loop average.

(2) we have given a first analytical solution representing circular magnetic monopole loops which goes through a pair of merons (with a unit topological charge) with non-trivial linking with the Wilson surface Σ.
This is achieved by solving the reduction differential equation for the adjoint color (magnetic monopole) field in the two–meron background field.



We do not need to use the Abelian projection to define magnetic monopoles in Yang-Mills theory!

Our analytical solution corresponds to a numerical solution found on a lattice in [Montero and J.W. Negele, hep-lat/0202023, Phys.Lett.B533, 322-329 (2002).]



We have not yet obtained the analytic solution representing magnetic loops in the 2-instanton background which were found in the numerical way in [Reinhardt & Tok, hep-th/0011068, Phys.Lett.B**505**, 131–140 (2001)]

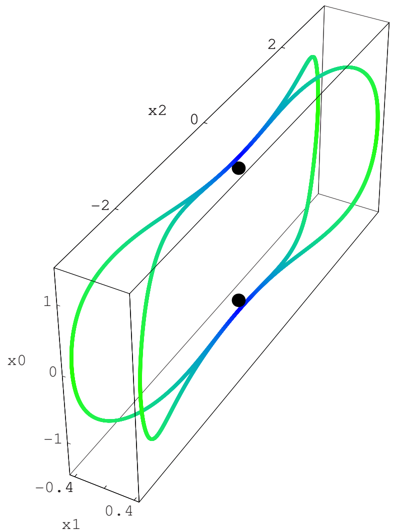


Figure 12: Plot of the two magnetic monopole loops for the gauge potential (**??**) projected onto the $x_1 - x_2 - x_0$-space (dropping the x_3-component). Rotations with angle π around the x_1- , x_2- and x_3-axis interchange the different monopole branches. The thick dots show the positions of the instantons.

Conjecture: A meron pair is the most relevant quark confiner in the original Yang-Mills theory, as Callan, Dashen and Gross suggested long ago.

dual Yang-Mills: magnetic monopole loops $\Longleftrightarrow$ original Yang-Mills: merons

⊙ Subjects to be investigated:

• Extending our results to SU(3):

· Continuum formulation
[K.-I. K., arXiv:0801.1274, Phys. Rev. D **77**, 085029 (2008)]
[K.-I. K., Shinohara & Murakami, 0803.0176, Prog. Theor. Phys. **120**, 1–50 (2008)]
For the Wilson loop in the fundamental rep.,

$$\boldsymbol{n} \in SU(3)/U(2) \neq SU(3)/[U(1) \times U(1)]$$

Quarks in the fundamental rep. can be confined by a non-Abelian magnetic monopole described by a single color field for any N in $SU(N)$ against the Abelian projection scenario.

· Lattice formulation [K-I.K., Shibata, Shinohara, Murakami, Kato and Ito, arXiv:0803.2451 [hep-lat]]
Preliminary numerical simulations e-Print: arXiv:0810.0956 [hep-lat] (Lattice 2008)

• Relationship between other topological objects: For gauge-invariant vortices equivalent to center vortices,
[K.-I. K., arXiv:0802.3829 [hep-th], J. Phys. G: Nucl. Part. Phys. **35**, 085001 (2008)]

• Clarifying the role of elliptic solutions interpolating dimeron and one-instanton:
Cervero, Jacobs & Nohl (1977). one-parameter family of solutions,
k=0: meron, k=1: instanton
dissociation of an instanton into two merons?

• Derive magnetic monopole loops in the multi-instanton background found in

Reinhardt and Tok, [hep-th/0011068], Phys.Lett.B **505**, 131–140 (2001).

• Obtaining the integration measure for collective coordinates:
of circular magnetic monopole loops

• Considering the relationship with the Gribov problem:
non-trivial Coulomb gauge vacua with $Q_P = \pm 1/2$

• D-brane intepretation: D-0 brane $\leftrightarrow$ meron

Drukker, Gross and Itzhaki, [hep-th/0004131], Phys.Rev.D**62**,086007 (2000).

• Evaluating the Wilson loop average from the linking number

• color confinement

It is desirable to make clear the relationship between color confinement in general and quark confinement based on dual superconductor picture. Our approach opens a path to investigate this issue, since we have recovered color symmetry in this approach of deriving the dual superconductor picture.

• Chiral symmetry breaking

The massive X_μ gluon exchange between quarks leads to an effective four-fermion interactions. This could lead to the spontaneous chiral symmetry breaking, just as in the Nambu–Jona-Lasinio model.

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Thank you for your attention!