Breaking of replica symmetry
— Another viewpoint of its emergence—

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Introduction

Symmetry in disordered systems are often hidden.
Replica Symmetry Breaking(RSB) — mainly discussed in mean-field spin glass model.

ex) Sherrington-Kirkpatrick model(fully connected spin-glass)

$$H = -\sum_{i<j}^{N} J_{ij} S_i S_j, \quad S_i = \pm 1, J_{ij} \sim N(0, 1/N)$$

The interaction causes

- frustration
- randomness

The Replica Method

$E[\cdots]$ — expectation w.r.t. the randomness.(J_{ij})

Identity(unfamiliar form)

$$E[\log Z] = \lim_{n\to 0} \frac{1}{n} \log E[Z^n]$$

1. Calculate $E[Z^n]$ for $n = 1, 2, 3, \cdots$.

$$\begin{aligned} E[Z^n] &= \mathrm{Tr}_1 \cdots \mathrm{Tr}_n E[\exp(-\beta \sum_{\alpha=1}^{n} H(\mathbf{S}^\alpha))] \\ &= \mathrm{Tr}_1 \cdots \mathrm{Tr}_n \exp(-\mathcal{H}_{eff}(\mathbf{S}^1, \cdots, \mathbf{S}^n)) \end{aligned}$$

2. Continuate the result to $n \in \mathbf{R}$

One can expect that all replicas of the system($\mathbf{S}^\alpha$) are equivalent...
Replica symmetry(RS)
$\to$ But it breaks in the low temperature!(instability)

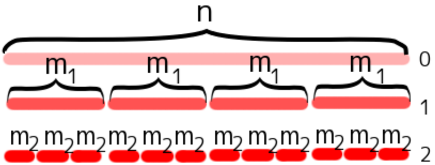
proposed symmetry breaking

Replica Symmetry Breaking(RSB) is introduced in a recursive manner.

$I := \{1, 2, 3 \cdots, n\}$: set of replica index

1. Divide the set I into n/m sets I_j with cardinality m.
2. Assume these two symmetries under permutation.
   1. permute indeces inside I_j.
   2. permute blocks I_j itself.
3. If this saddle point is still unstable, go back to 1 with setting $I = I_j$ and break them in the same way.

If we have to break the replicas into K hierarchy, the system is called K-RSB system.



conventional analysis

ex)mean field Potts glass model

$$H = -\sum_{i<j} J_{ij}\delta_{\sigma_i,\sigma_j}, \quad \sigma_i = 1, 2, \cdots, p, \;\; J_{ij} \sim N(0, 1/N)$$

temperature	type of RSB
$T_{c1} < T$	RS holds
$T_{c2} < T < T_{c1}$	RS is instablized $\Rightarrow$ 1RSB
$T < T_{c2}$	1RSB is instablized $\Rightarrow$ ∞-RSB

It seems that 1RSB is differnt from K-RSB($K \geq 2$)...

type of RSB

Quite generally, systems often belong to one of the below three systems.

- RS — no hierarchies (or one-level hierarchy)
- 1RSB — two-level hierarchy
- ∞-RSB — infinite-level hierarchy

Question

Why do we rarely observe K-RSB ($2 \leq K < \infty$)?
or
Is there any source of universality for 1RSB?

reconsideration on the replica method

What we really do in the replica calculation is...

$$\lim_{N\to\infty} \frac{1}{N} E[\log Z] = \lim_{n\to 0} \lim_{N\to\infty} \frac{1}{Nn} \log E[Z^n]$$

1. Taking thermodynamic limit(saddle-point method).
2. Performing “analytic” continuation of n.

a general lesson from statistical mechanics

external field $h \to 0$ limit after thermodynamic limit can cause a lack of “analiticity”, or “symmetry breaking.”

$\Rightarrow$ Phase transition w.r.t. n may be significant for RSB!

statistical mechanical setup

one-parameter/pseudo-temperature extention of the free energy

$$E[\log Z] = \lim_{n\to 0} \frac{1}{n} \log E[Z^n] = \lim_{n\to 0} \frac{1}{n} \log E[\exp(-\beta n F)]$$

- $E[\cdots]$ — Tr
- F — Hamiltonian
- $1/(\beta n)$ — temperature

Canonical statistical mechanics!

RS ansatz is correct for $n \gg 1$ (*i.e.* at low temperature in $1/\beta n$).
Q. How RS is broken when lowering $n \to 0$?
$\Rightarrow$ Of course, one possibility is instability, but is there another?

Thermodynamics of the replica number n

There is thermodynamic constraints to the continuation of n.
Def.

$$\phi(n) := -\frac{1}{\beta n} \log E[Z^n]$$

1. Energy and entropy is monotone increasing function of the temperature.
(monotonicity)

$$\frac{\mathrm{d}}{\mathrm{d}n}\phi(n) \leq 0$$

2. Heat capacity is non-negative.
(convexity)

$$\frac{\mathrm{d}^2}{\mathrm{d}n^2}(n\phi(n)) \leq 0$$

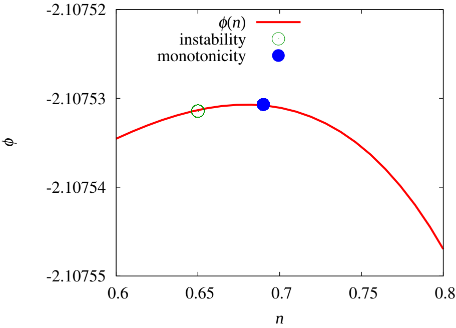
These hold for all models.

$T_{c_2} < T < T_{c1}$

mean field Potts glass model

$$H = -\sum_{i<j} J_{ij}\delta_{\sigma_i,\sigma_j}, \quad \sigma_i = 1, 2, \cdots, p, \ \ J_{ij} \sim N(0, 1/N)$$

When decreasing n, monotonicity is broken before instability occurs.
⇒ Monotonicity is relevant at $T_{c2} < T < T_{c1}$!

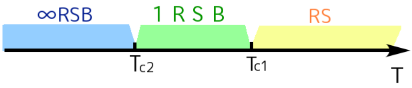


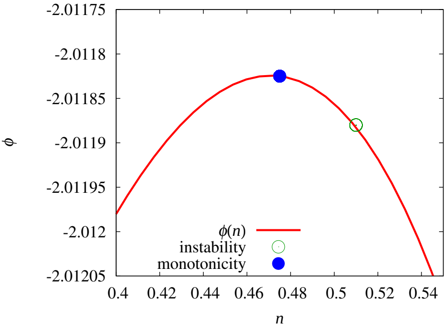
$p = 4$

$T < T_{c2}$

mean field Potts glass model

$$H = -\sum_{i<j} J_{ij}\delta_{\sigma_i,\sigma_j}, \quad \sigma_i = 1, 2, \cdots, p, \;\; J_{ij} \sim N(0, 1/N)$$



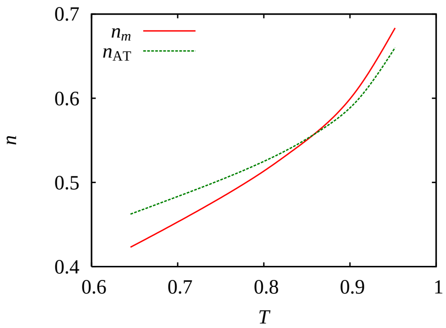


When decreasing n, instability occurs before before monotonicity is broken.

$\Rightarrow$ Instability is relevant at $T < T_{c2}$!

$p = 4$

“phase diagram”



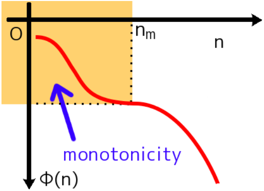
Relevance of instability and monotonicity is switched at T_{c2}.

useful proposition

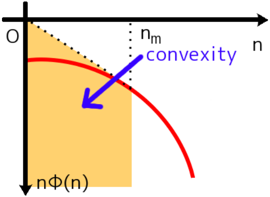
proposition

If $\phi'(n_m) = 0$ for $n_m > 0$, then $\phi(n) = \phi(n_m)$ for $0 < n < n_m$.

from the monotonicity,



from the convexity,



Both are satisfied at the same time only when $\phi(n) = \phi(0)$ for $0 < n < n_m$.

Rem. This proposition is valid for all models.

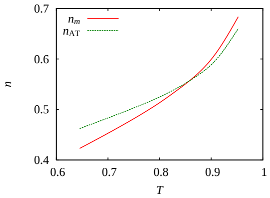
1RSB solution is derived!

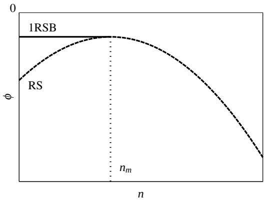
Using this theorem at $T_{c2} < T < T_{c1}$, we get

$$E[F] = \phi(n_m) \quad \text{where } \phi'(n_m) = 0.$$

Relation to 1RSB free energy

- n_m
  $\longrightarrow$ 1RSB block size m
- $\phi'(n_m) = 0$
  $\longrightarrow$ saddle point equation of m
- monotonicity condition
  $\longrightarrow$ saddle-point condition





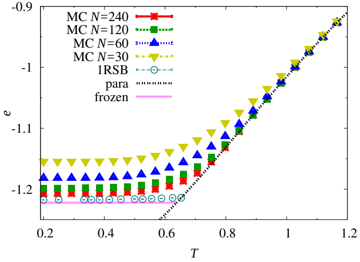
Summary 1 — What did we do?

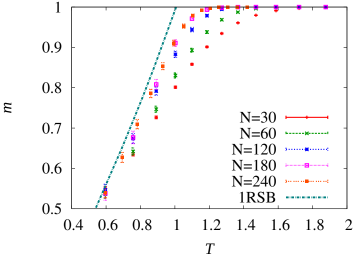
- We perform the continuation w.r.t. n **without** conventional RSB scheme, but get the same result as 1RSB.
- The continuation is performed by a macroscopic condition, which is related to the thermodynamics.
- Thermodynamics of finite replica number is universal feature for the replicated systems.

We can conclude that...
1RSB is actually universal because of the **thermodynamics**!

Summary 2 — What can we do using this?

It is possible to construct 1RSB free energy if conventional RSB theory does not work.
eg) spin-glass model on a random graph





This framework shold be applied to the finite-dimensional cases....