Randomness in Infinitesimal Extent in the Numerical Implementation of the McLerran-Venugopalan Model

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Things to be discussed

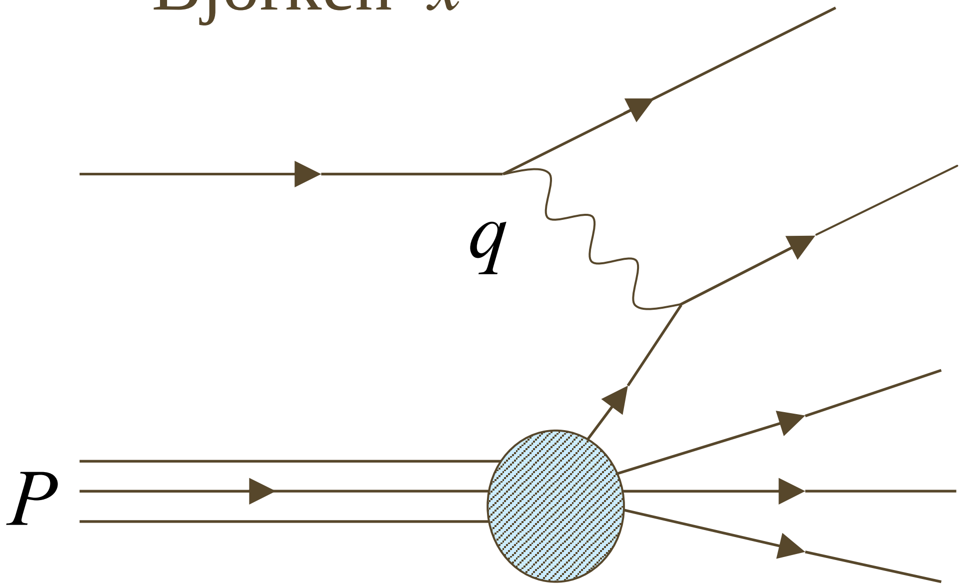
- What is the MV model?
- Why is it so hard analytically?
- How is it solved numerically?
- What is the problem?
- How much difference results from that?
- Conclusions
  - Venugopalan, Krasnitz, Nara, Lappi ... suspicious!
  - Venugopalan, Romatchke, Lappi ... suspicious!!

General Introduction

- Small-x
- Parton Saturation
- Color Glass Condensate
- McLerran-Venugopalan Model

Frequently Used Variables

- Two Variables (in DIS)
  - Virtuality Q^2
  - Bjorken x



$$Q^2 = -q^2$$

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{s + Q^2 + M^2}$$

small-x $x \ll 1$

⇕

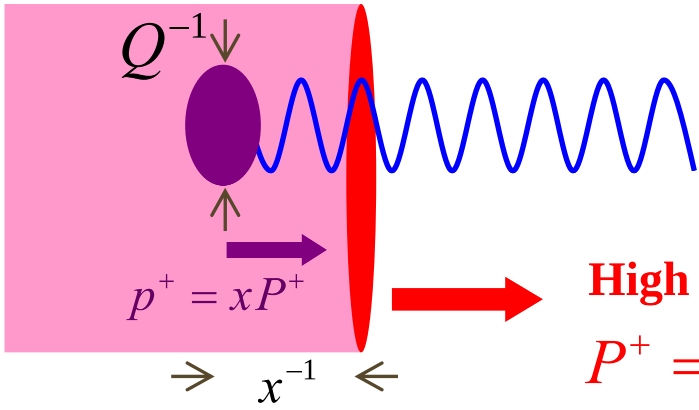
high energy $s \gg Q^2$

Convenient Interpretation

- Intuitive Meaning of Two Variables
  - Transverse Momentum Q^2

    Transverse size of partons
  - Bjorken x

    Longitudinal fraction of parton momentum



Light Projectile (photon if DIS)

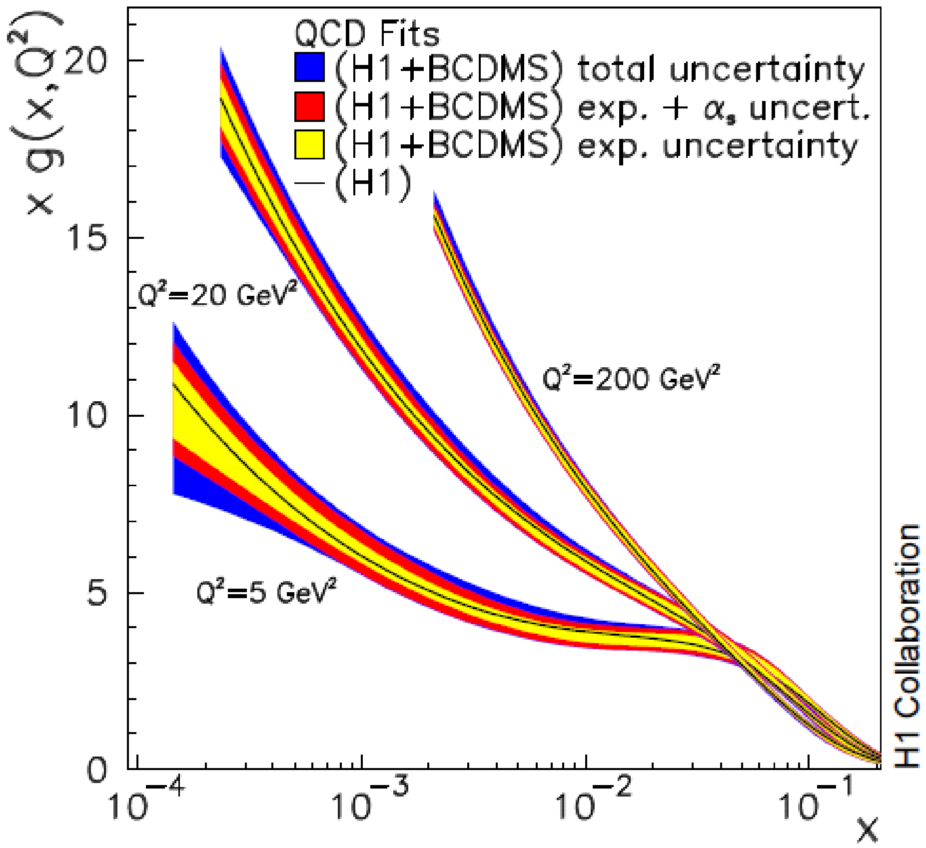
$$q^\mu = (q_0, q_\perp, 0)$$

High Energy Hadron Target

$$P^+ = \tfrac{1}{\sqrt{2}}(P^0 + P^z) \sim \infty$$

Gluon Evolution

- Parton (Gluon) distribution grows up



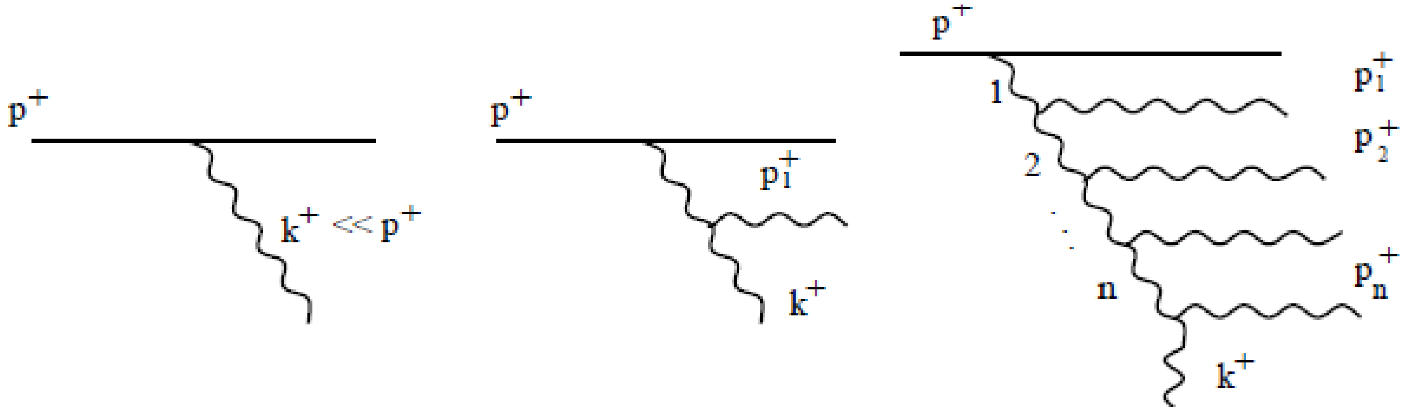
as x goes smaller
BFKL dynamics

or

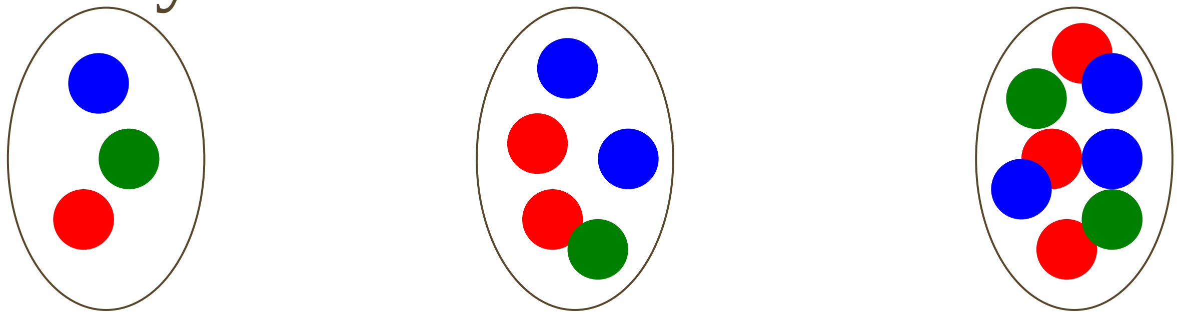
as Q goes larger
DGLAP dynamics

Going to smaller x with fixed Q

- Gluon increases with a fixed transverse area



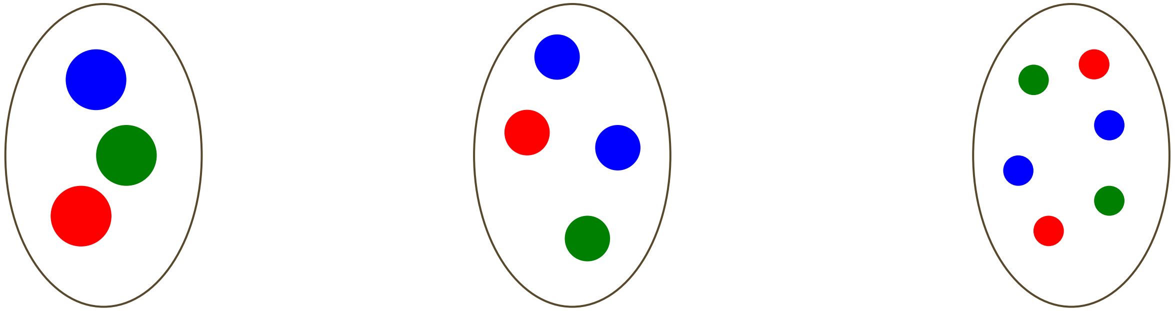
- Graphically



small-*x* → Dense Gluon Matter

Going to larger Q^2 with fixed x

- Gluon slowly increases with a decreasing area
- Graphically, in the same way,



large *Q* → Dilute Gluon Matter

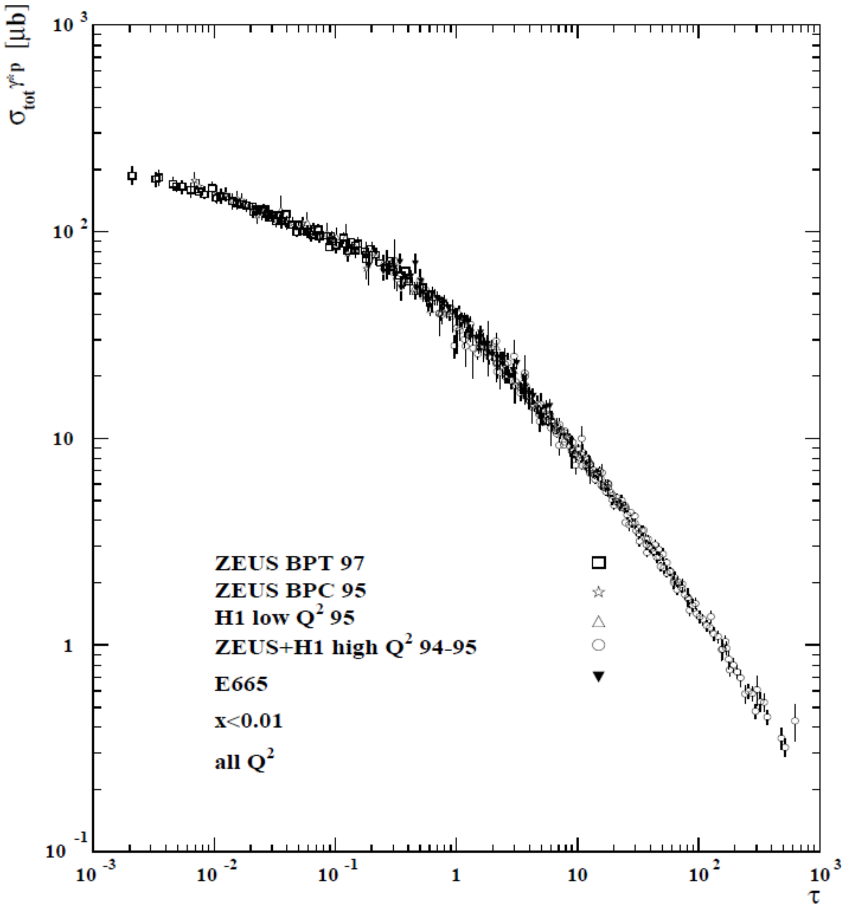
- When does the distribution come to overlap?

$$\frac{xg(Q_s, x)}{(N_c^2 - 1)\cdot Q_s^2 \cdot \pi R_A^2} \sim 1$$

Gluons with $k_t << Q_s(x)$ are saturated.

Saturated, then, Simple!

- x and Q not independent but...



$$\sigma_{\gamma^* p}(x,Q^2) \rightarrow \sigma_{\gamma^* p}(Q^2/Q_s^2(x))$$

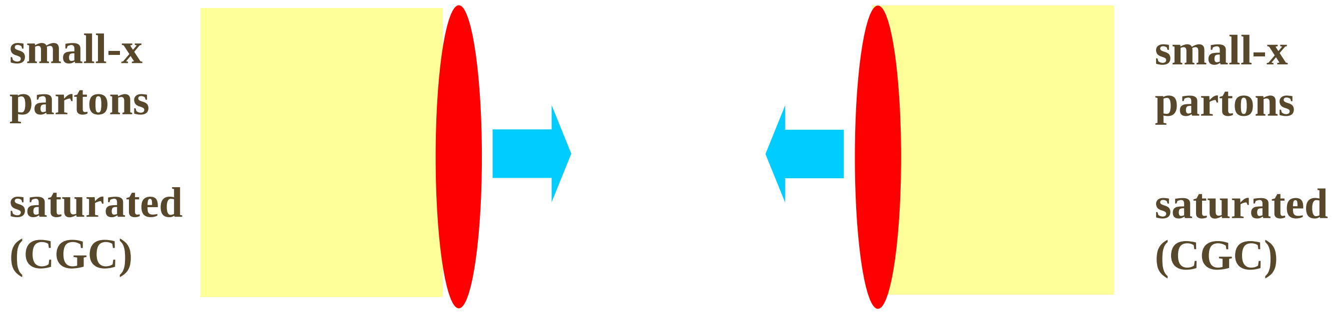
$$Q_s^2(x) = Q_0^2(x/x_0)^{-\lambda}$$

Called the Geometric Scaling

Stasto-Golec-Biernat-Kwiecinski Plot

Nucleus-Nucleus Collisions

- Particles $p_t < 1\text{GeV}$



$$x \sim p_t / \sqrt{s} \sim 10^{-2} \quad (\sqrt{s} = 200\text{GeV})$$

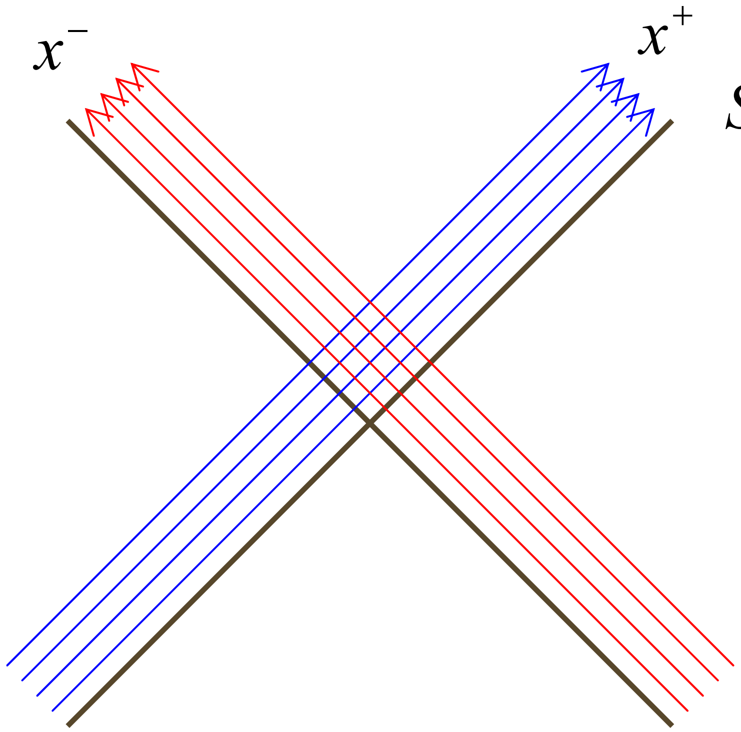
$$Q_s = Q_0 (x_0 / x)^{\lambda} \cdot A^{1/6} \sim 1\text{-}2\,\text{GeV}$$

Initial Time-Evolution scales as $\sim \tau Q_s$
Initial Energy-Density proportional to $Q_s{}^4$

Fries-Kapusta-Li ('06)
K.F. ('07)

Dense-Dense Scattering

- Scattering amplitude in the Eikonal approx.



Dense Target **Dense Projectile**

$$S \sim \left\langle \sum_{\{\rho_t\}} \mathcal{W}_x[\rho_t] \prod_{\{\rho_t\}} W \cdot \sum_{\{\rho_p\}} \mathcal{W}_{x'}[\rho_p] \prod_{\{\rho_p\}} V \right\rangle$$

$$W(x_\perp) = \exp\left[ig \int dz^+ A^-(z^+, x_\perp) \right]$$

$$V(x_\perp) = \exp\left[ig \int dz^- A^+(z^-, x_\perp) \right]$$

Classical Approximation

- Stationary-point approximation

$$S \sim \left\langle \sum_{\{\rho_t\}} \mathcal{W}_x[\rho_t] \prod_{\{\rho_t\}} W \cdot \sum_{\{\rho_p\}} \mathcal{W}'_{x'}[\rho_p] \prod_{\{\rho_p\}} V \right\rangle$$

$$= \sum_{\{\rho_t,\rho_p\}} \mathcal{W}_x[\rho_t]\mathcal{W}'_{x'}[\rho_p]\int[\mathcal{D}A]\exp\left[iS_{\mathrm{YM}} + iS_{\mathrm{source}}[\rho_t,\rho_p,W,V]\right]$$

$$\sim \sum_{\{\rho_t,\rho_p\}} \mathcal{W}_x[\rho_t]\mathcal{W}'_{x'}[\rho_p]\int[\mathcal{D}A]\exp\left[iS_{YM} - i\int d^4x\,(\rho_t^a A_a^- + \rho_p^a A_a^+)\right]$$

Stationary-point approx. is made at $\left.\dfrac{\partial S_{\mathrm{YM}}}{\partial A_a^\mu}\right|_{A=\bar{A}} = \delta^{\mu-}\rho_t^a + \delta^{\mu+}\rho_p^a$

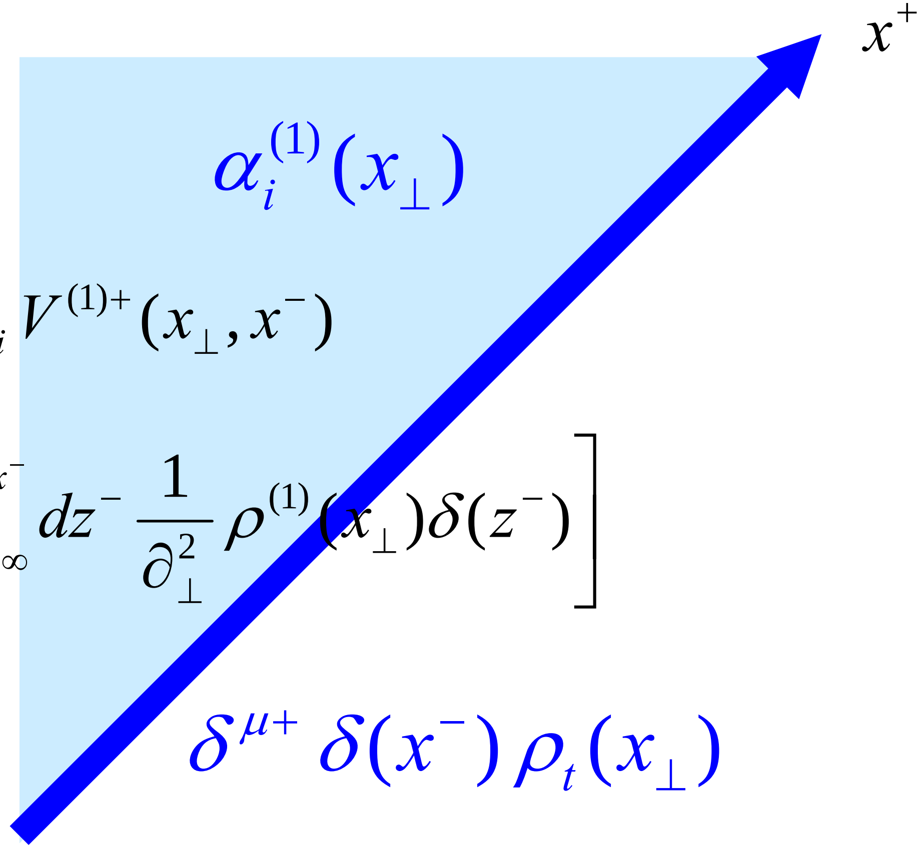
Solutions

- One-source problem is solvable

$$\mathcal{A}^+ = \mathcal{A}^- = 0$$

$$\mathcal{A}_i = \alpha_i^{(1)} = -\frac{1}{ig} V^{(1)}(x_\perp, x^-)\,\partial_i V^{(1)+}(x_\perp, x^-)$$

$$V^{(1)+}(x_\perp, x^-) = P_{x^-}\exp\left[-ig\int_{-\infty}^{x^-} dz^- \frac{1}{\partial_\perp^2}\rho^{(1)}(x_\perp)\delta(z^-)\right]$$



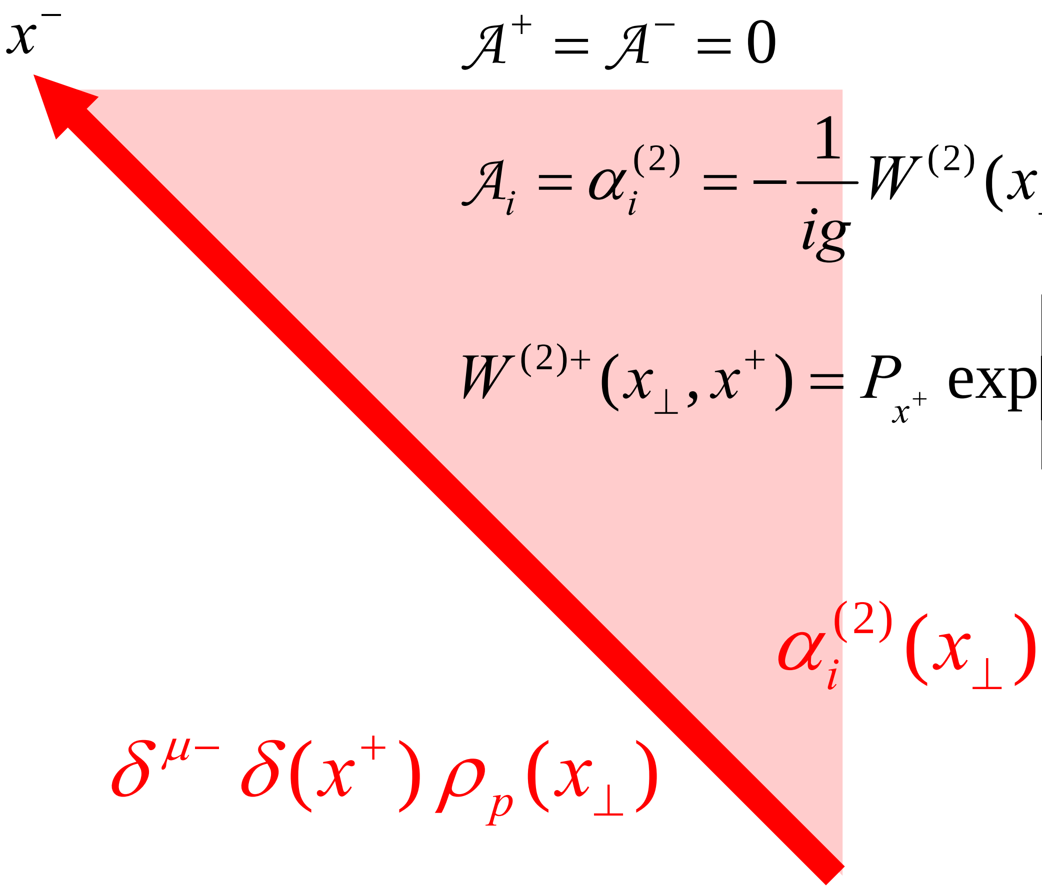
Solutions

- Likewise

$$\mathcal{A}^+ = \mathcal{A}^- = 0$$

$$\mathcal{A}_i = \alpha_i^{(2)} = -\frac{1}{ig} W^{(2)}(x_\perp, x^+)\,\partial_i W^{(2)+}(x_\perp, x^+)$$

$$W^{(2)+}(x_\perp, x^+) = P_{x^+} \exp\left[-ig\int_{-\infty}^{x^+} dz^+ \frac{1}{\partial_\perp^2}\rho^{(2)}(x_\perp)\delta(z^+)\right]$$



Boundary Conditions

- Two-source problem

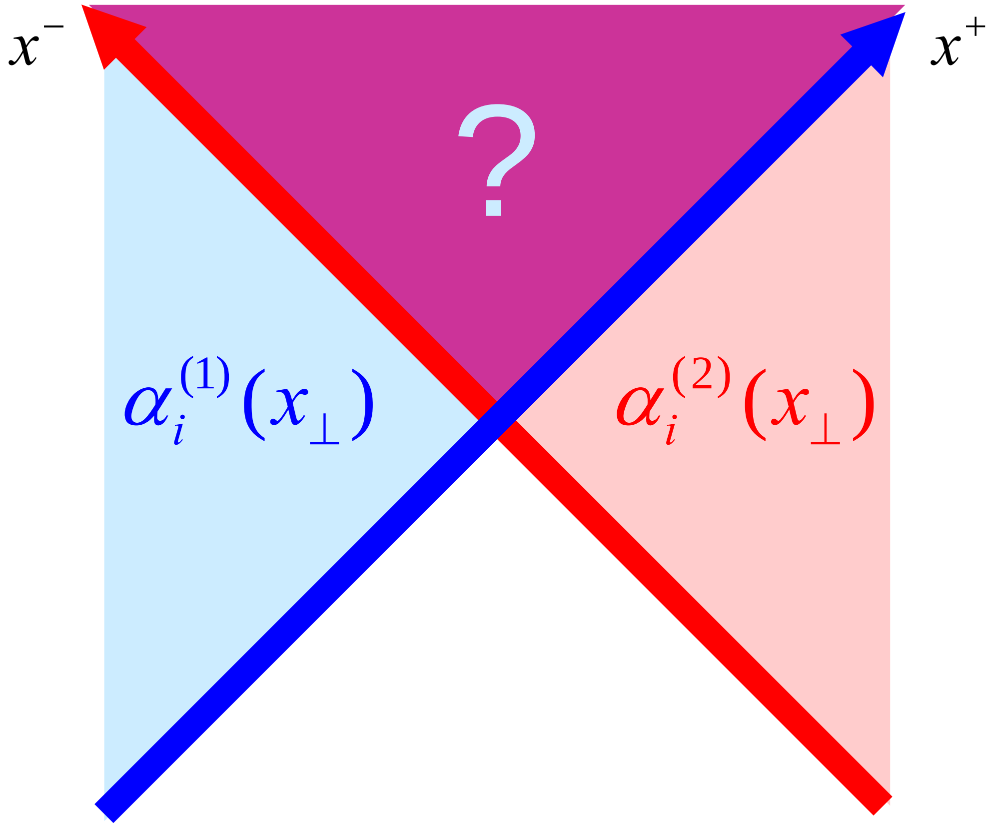
On the light cone $(\tau = 0)$

$$\bar{A}_i = \alpha_i^{(1)} + \alpha_i^{(2)}$$

$$\bar{A}_\eta = 0$$

$$\bar{E}^i = 0$$

$$\bar{E}^\eta = ig[\alpha_i^{(1)}, \alpha_i^{(2)}]$$



McLerran-Venugopalan Model

- Gaussian weight

$$\mathcal{W}_x[\rho] = \exp\left[-\int d^3x \frac{|\rho(x)|^2}{2\mu_x^2}\right]$$

μ_x is related to $Q_s(x)$

larger μ = larger ρ = dense gluons = larger Q_s

- Once A is known, observables like the field energies are calculable in unit of μ.

$$\varepsilon = \left\langle\left\langle E^2 + B^2 \right\rangle\right\rangle_{\rho_t,\rho_p}$$

- Two steps: solve $A[\rho_t,\rho_p]$ and take $\left\langle\left\langle \cdots \right\rangle\right\rangle_{\rho_t,\rho_p}$

Expectation Values

- Physical observables as a function of

$$V^{(1)+}(x_\perp, x^-) = P_{x^-} \exp\left[-ig \int_{-\infty}^{x^-} dz^- \frac{1}{\partial_\perp^2} \rho^{(1)}(x_\perp)\delta(z^-) \right]$$

$$W^{(2)+}(x_\perp, x^+) = P_{x^+} \exp\left[-ig \int_{-\infty}^{x^+} dz^+ \frac{1}{\partial_\perp^2} \rho^{(2)}(x_\perp)\delta(z^+) \right]$$

- Gaussian weight

$$\mathcal{W}[\rho] = \exp\left[-\int d^2 x_T dx^\pm \frac{|\rho(x)|^2}{2g^2 \mu^2(x^\pm)} \right]$$

Numerical Method

- Approximation

$$\overline{V}^{(1)+}(x_\perp) = \exp\left[-ig\,\theta(x^-)\frac{1}{\partial_\perp^2}\rho^{(1)}(x_\perp) \right]$$

$$\overline{W}^{(2)+}(x_\perp) = \exp\left[-ig\,\theta(x^+)\frac{1}{\partial_\perp^2}\rho^{(2)}(x_\perp) \right]$$

- Gaussian weight

$$\overline{\mathcal{W}}[\rho] = \exp\left[-\int d^2x_T \frac{|\rho(x)|^2}{2g^2\overline{\mu}^2} \right] \qquad \overline{\mu}^2 = \int dx^\pm \mu^2(x^\pm)$$

Delta Functions

- One Dirac delta function

$$V^{+}(x_{\perp},x^{-}) = P_{x^{-}} \exp\left[-ig\int_{-\infty}^{x^{-}} dz^{-} \frac{1}{\partial_{\perp}^{2}} \rho(x_{\perp})\delta(z^{-}) \right]$$

- Another Dirac delta function

$$\left\langle \rho_a(x_T,x^{-})\rho_b(y_T,y^{-}) \right\rangle = g^2\mu^2(x^{-})\delta^{(2)}(x_T - y_T)\delta(x^{-} - y^{-})$$

- Need for appropriate regularization

Regularized Expressions

- Longitudinal Extent

$$V_\varepsilon^+(x_\perp, x^-) = P_{x^-} \exp\left[-ig\int_{-\infty}^{x^-} dz^- \frac{1}{\partial_\perp^2} \rho_\varepsilon(x_\perp, z^-) \right]$$

- Randomness

$$\left\langle \rho_a(x_T, x^-) \rho_b(y_T, y^-) \right\rangle_\varsigma = g^2 \mu^2(x^-) \delta^{(2)}(x_T - y_T) \delta_\varsigma(x^- - y^-)$$

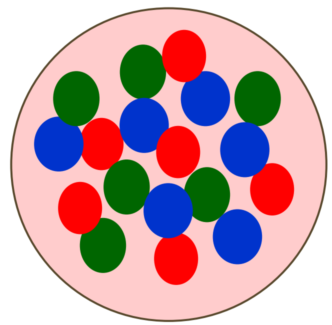
- Limit

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(x_T, x^-) = \rho(x_T)\delta(x^-)$$

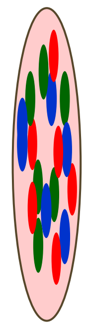
$$\lim_{\varsigma\to 0} \delta_\varsigma(x^-) = \delta(x^-)$$

Non-commutative Limits

- Numerical Method
  - Take the $\varepsilon \to 0$ limit first
  - Take the $\zeta \to 0$ limit then



- In reality, we should...
  - Take the $\zeta \to 0$ limit first
  - Take the $\varepsilon \to 0$ limit then



Example

- Tadpole Expectation Value

$$\left\langle V_\varepsilon^+(x_T)\right\rangle_\varsigma$$

- We know the analytical result

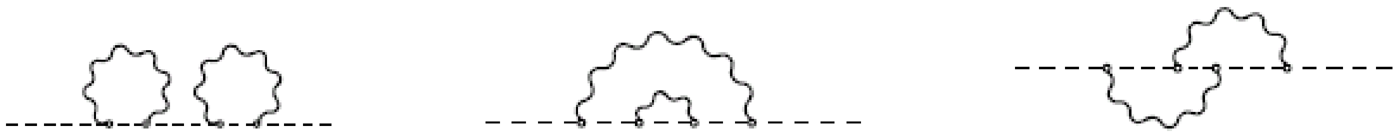
$$\lim_{\varepsilon\to 0}\lim_{\varsigma\to 0}\left\langle V_\varepsilon^+(x_T)\right\rangle_\varsigma = \exp\left[-\frac{N_c^2-1}{4N_c}g^4\overline{\mu}^2 L(0)\right]$$

$$L(0)=\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2)^2}\sim \# L^2 \qquad L:\text{number of lattice sites}$$

From Fukushima-Hidaka

$$U(b^-, a^-|\boldsymbol{x}_\perp) = \mathcal{P}\exp\left[-\mathrm{i}g^2 \int_{a^-}^{b^-} \mathrm{d}z^- \mathrm{d}^2\boldsymbol{z}_\perp\, G_0(\boldsymbol{x}_\perp - \boldsymbol{z}_\perp)\, \rho_a(z^-, \boldsymbol{z}_\perp)\, t^a\right]$$

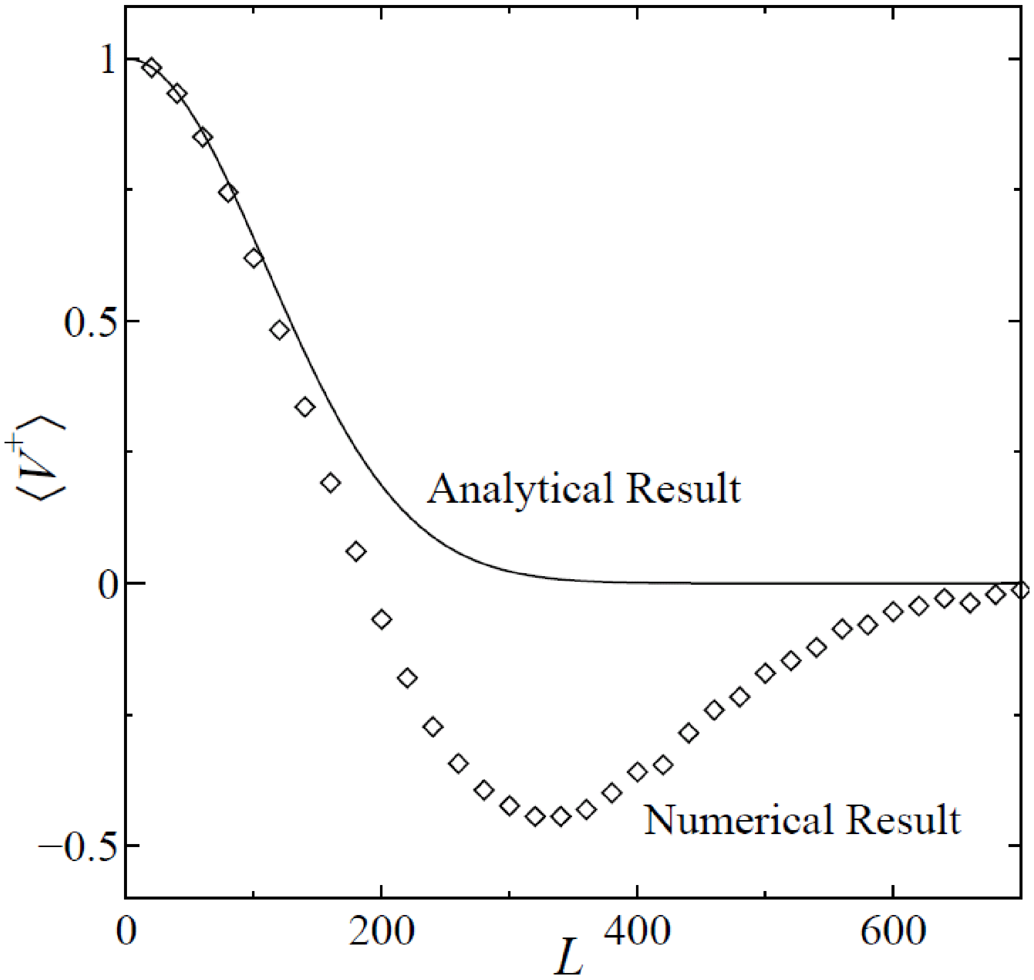
$$\langle U(b^-, a^-|\boldsymbol{x}_\perp)\rangle = \sum_{n=0}^{\infty}(-\mathrm{i}g^2)^n \int \prod_{i=1}^{n} \mathrm{d}^2\boldsymbol{z}_{i\perp}\, G_0(\boldsymbol{x}_\perp - \boldsymbol{z}_{i\perp}) \int_{a^-}^{b^-} \mathrm{d}z_1^- \int_{a^-}^{z_1^-} \mathrm{d}z_2^- \cdots \int_{a^-}^{z_{n-1}^-} \mathrm{d}z_n^- \times$$
$$\times \langle \rho_{a_1}(z_1^-, \boldsymbol{z}_{1\perp})\rho_{a_2}(z_2^-, \boldsymbol{z}_{2\perp}) \cdots \rho_{a_n}(z_n^-, \boldsymbol{z}_{n\perp})\rangle\, t^{a_1}\, t^{a_2} \cdots t^{a_n}\,. \qquad (2.7)$$



(a) (b) (c)

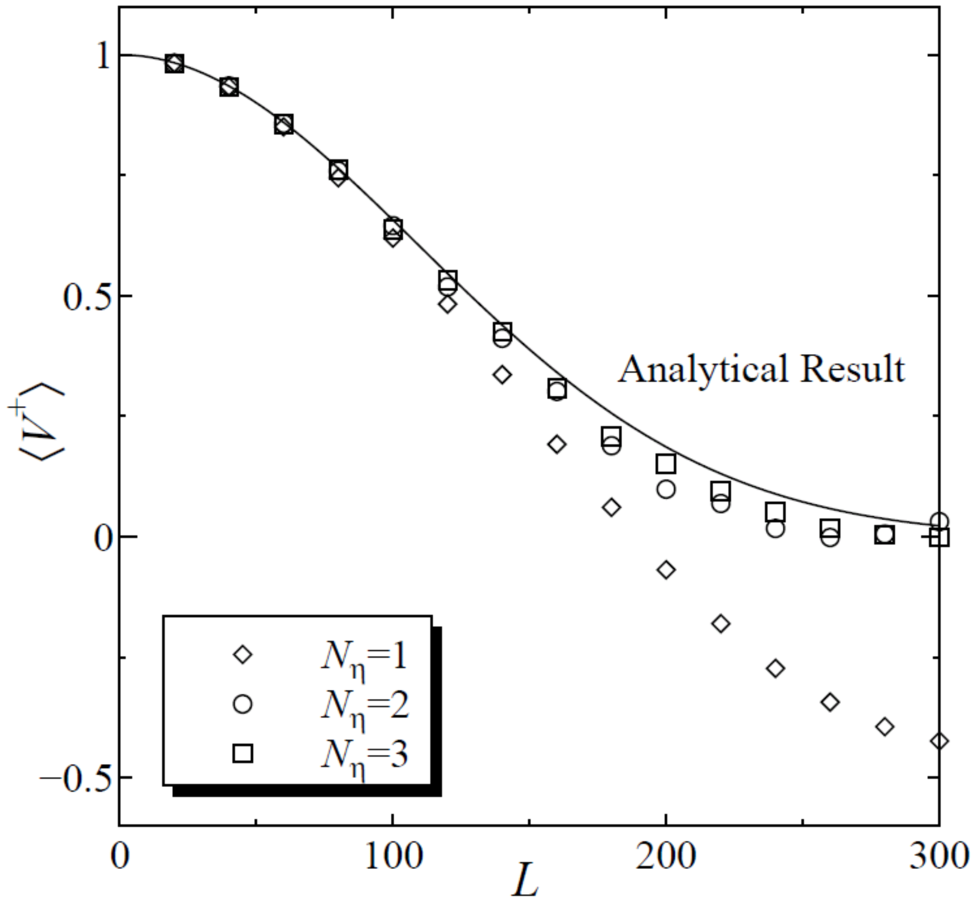
$$\langle U(b^-, a^-|\boldsymbol{x}_\perp)_{\beta\alpha}\rangle =$$
$$= \delta_{\beta\alpha} + (-\mathrm{i}g^2)^2 \int_{a^-}^{b^-} \mathrm{d}z_1^- \mathrm{d}^2\boldsymbol{z}_{1\perp} \int_{a^-}^{z_1^-} \mathrm{d}z_2^- \mathrm{d}^2\boldsymbol{z}_{2\perp}\, G_0(\boldsymbol{x}_\perp - \boldsymbol{z}_{1\perp}) G_0(\boldsymbol{x}_\perp - \boldsymbol{z}_{2\perp}) \times$$
$$\times \langle \rho_{a_1}(z_1^-, \boldsymbol{z}_{1\perp})\rho_{a_2}(z_2^-, \boldsymbol{z}_{2\perp})\rangle \left(t^{a_1} t^{a_2}\right)_{\beta\gamma} \langle U(z_2^-, a^-|\boldsymbol{x}_\perp)_{\gamma\alpha}\rangle$$
$$= \delta_{\beta\alpha} - \frac{g^4}{2} C_2(r)\, \delta_{\beta\gamma} \int \mathrm{d}^2\boldsymbol{z}_\perp\, G_0^2(\boldsymbol{x}_\perp - \boldsymbol{z}_\perp) \int_{a^-}^{b^-} \mathrm{d}z^- \mu^2(z^-) \langle U(z^-, a^-|\boldsymbol{x}_\perp)_{\gamma\alpha}\rangle\,,$$

SU(2) Tadpole

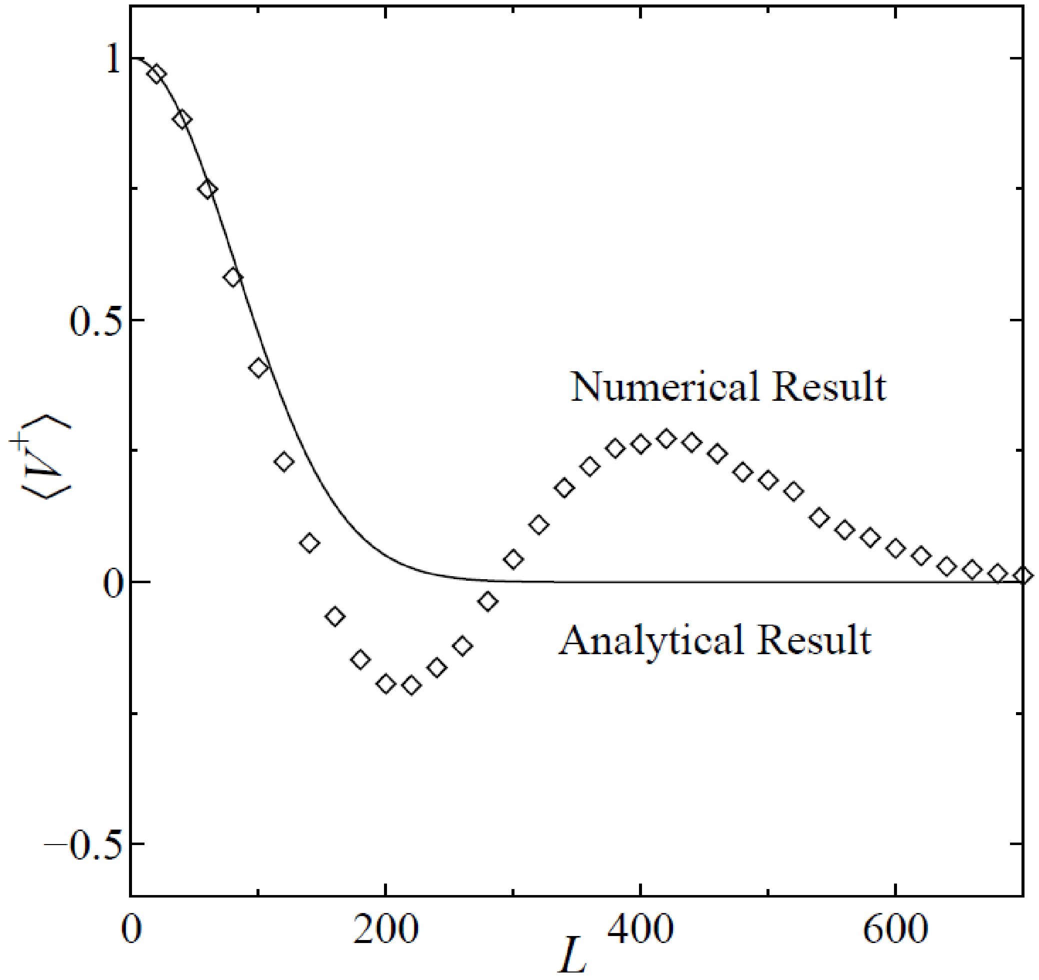


Improvement

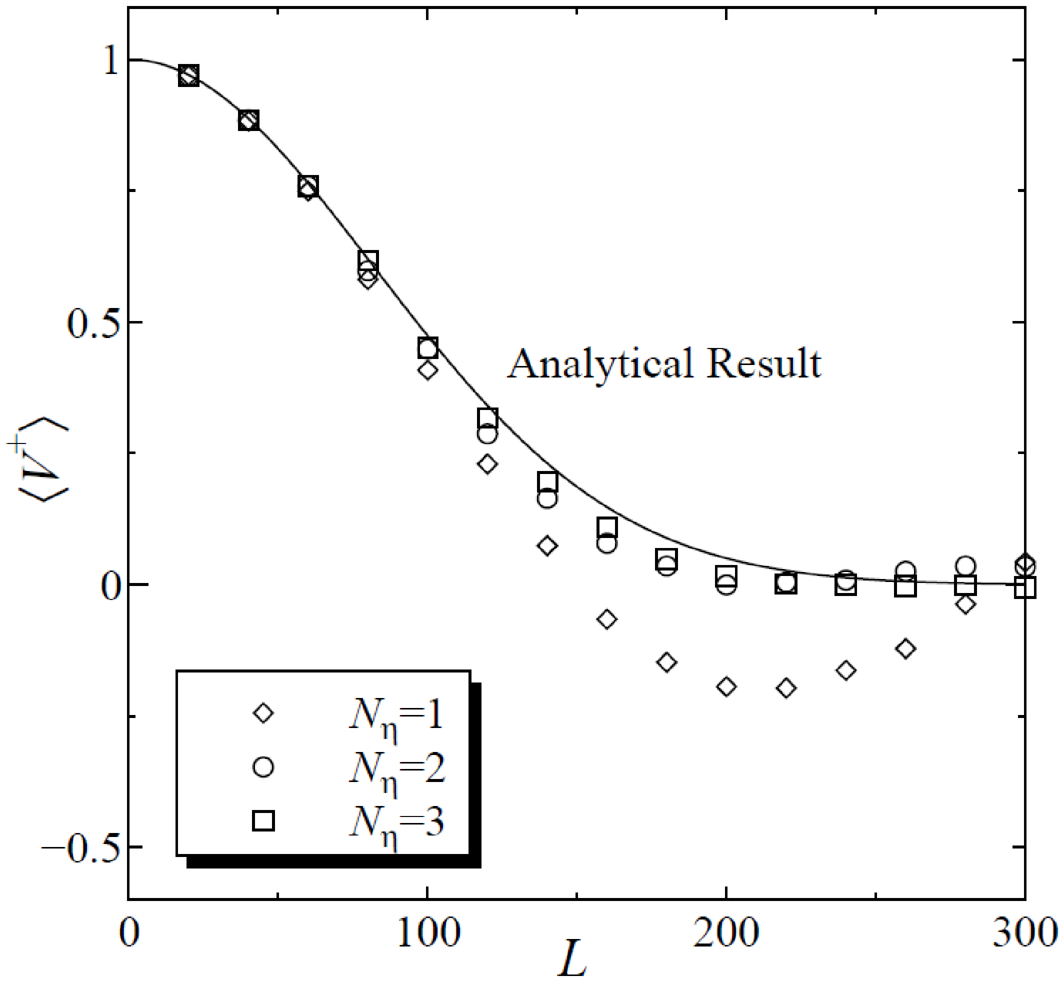
- Insert the random sheet



SU(3) Tadpole



Improvement



So far, So good... But!

- Field Strength Expectation Value

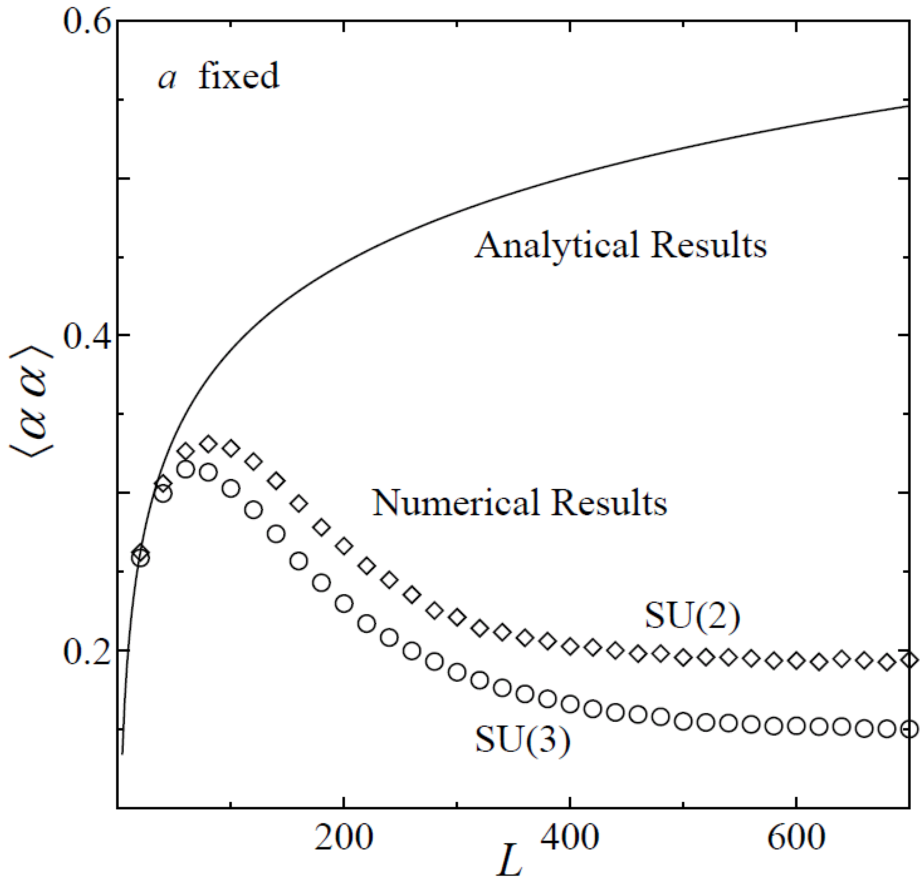
$$\left\langle \alpha_i^a \alpha_j^b \right\rangle = -\frac{1}{g^2}\left\langle (V\partial_i V^+)^a (V\partial_j V^+)^b \right\rangle = \delta^{ab}\delta_{ij} g^2 \overline{\mu}^2 \left\langle \alpha\alpha \right\rangle$$

- Amazingly, the analytical result is known

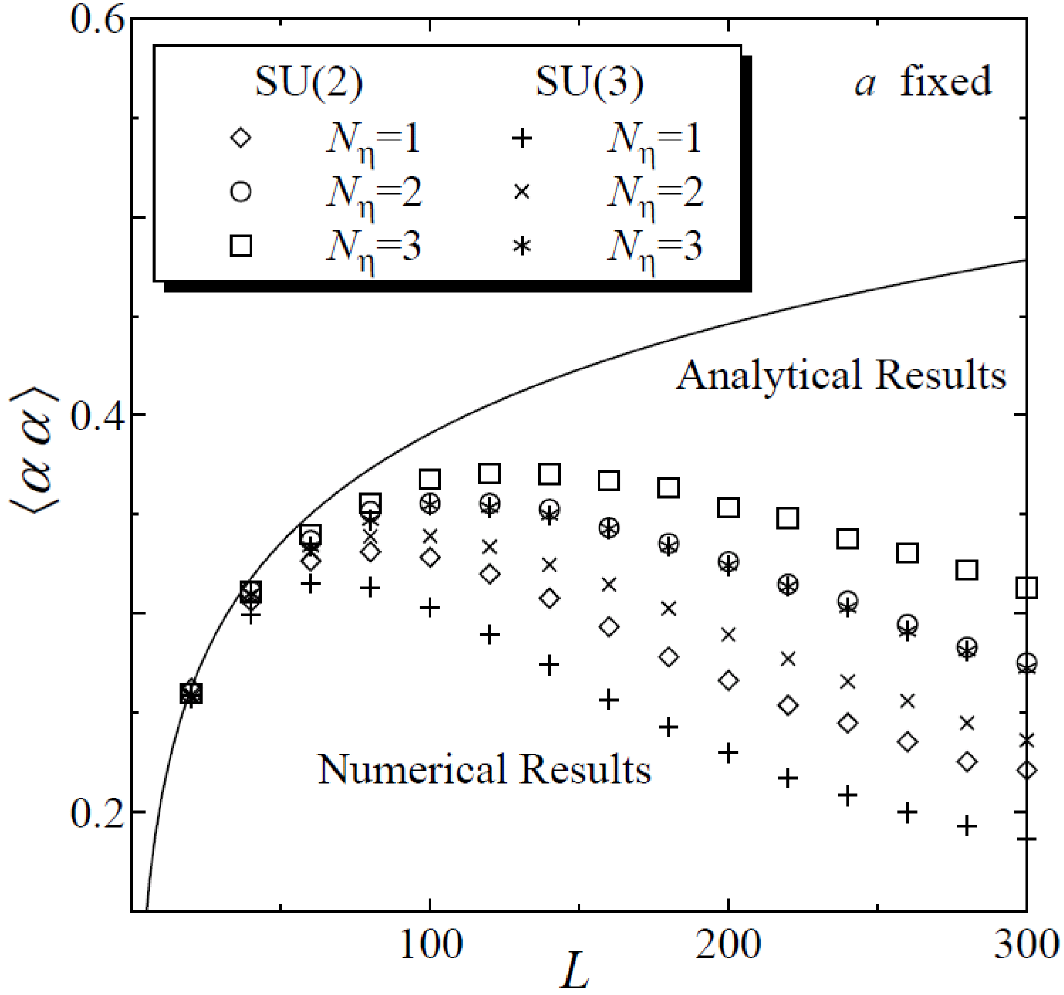
$$\left\langle \alpha\alpha \right\rangle = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2} \sim \# \ln L = \# \ln\left(\frac{R_A}{a}\right)$$

Infrared Dependence

- Dependence on the system size

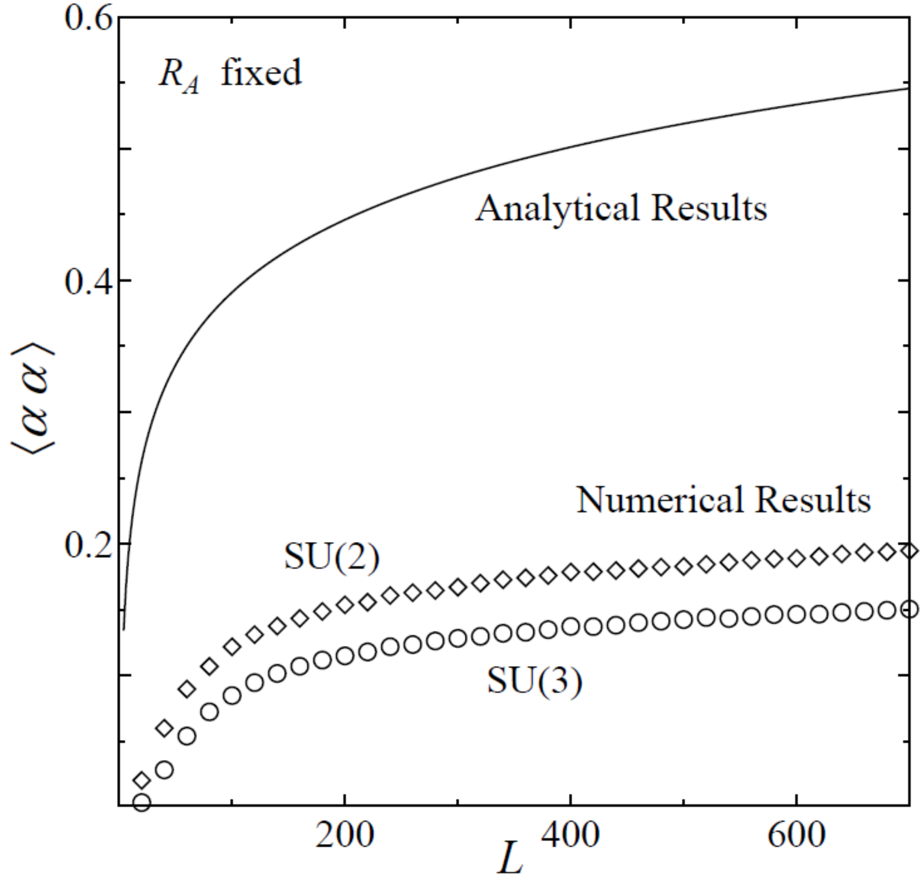


Improvement… slightly

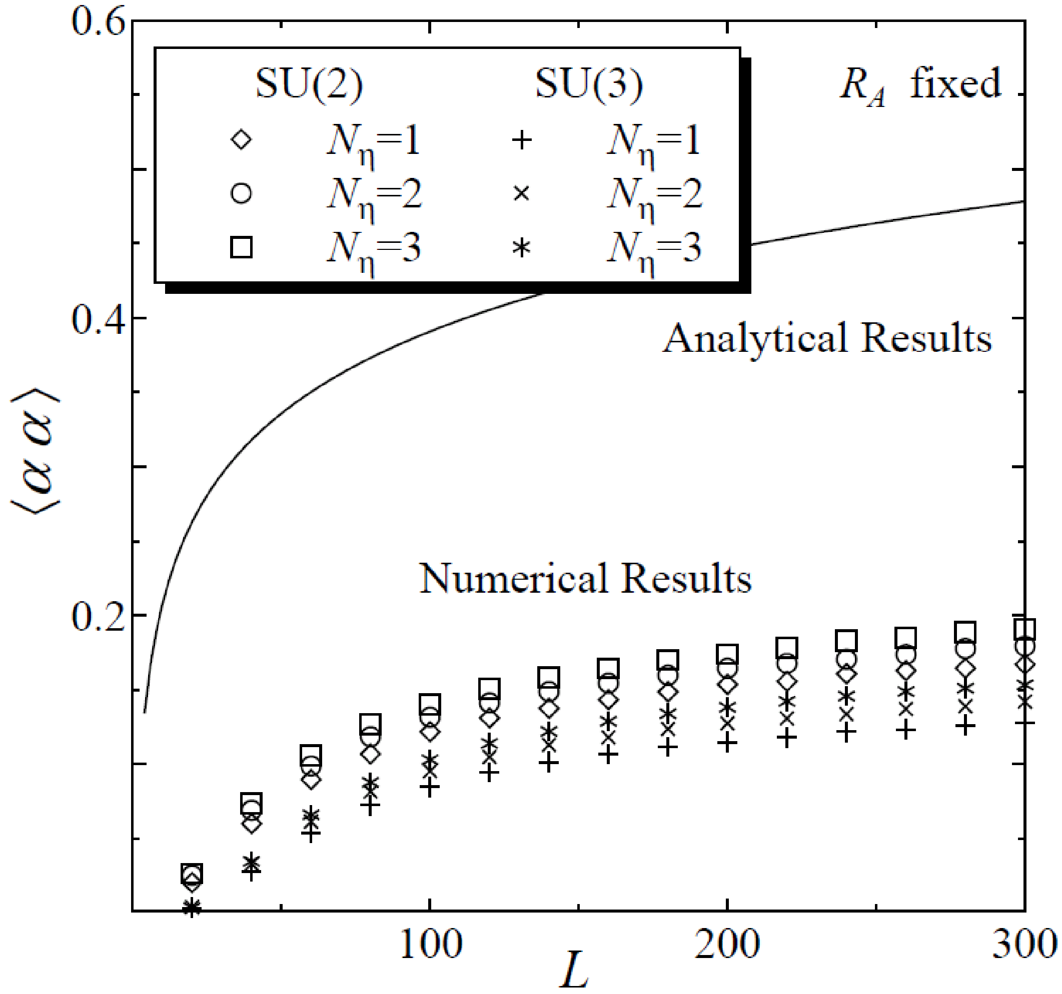


Ultraviolet Dependence

- Dependence on the lattice spacing

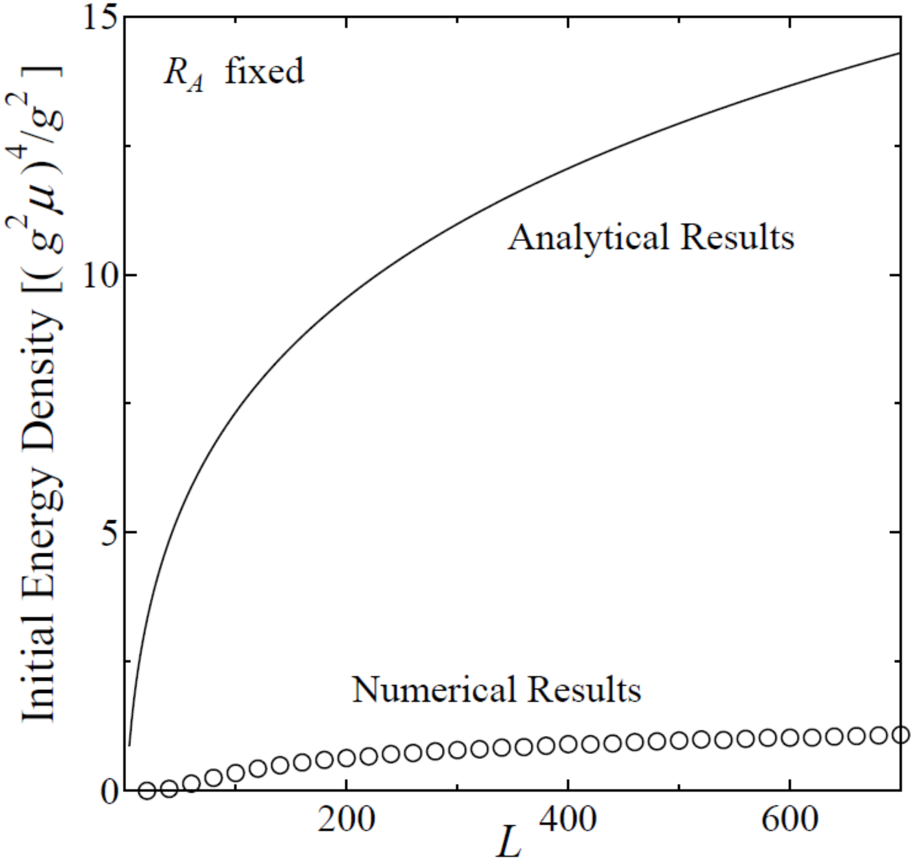


Improvement... only little



Initial Energy Density

- Analytical Formula $\varepsilon = 2N_c(N_c^2 - 1)\langle \alpha\alpha \rangle^2$



Discussions

- Numerical implementation of the MV model assumes an irrelevant order of two limits.
- Energy density underestimates smaller by ~16.
- Then, numerical calculations meaningless???
- Maybe... but μ could rescue them...
- If μ is twice larger, energy density becomes 16 times larger...

Conclusions

- Numerical simulations in the MV model are not equivalent to analytical calculations.
- Need for inclusion of the path-ordering in the longitudinal direction. Large cost...
- Because this is about the fine structure in the longitudinal (rapidity) direction, instability scenario might be significantly affected. There might not be instability at all!?