QCD thermodynamics with improved Wilson quarks at zero and finite densities

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**MOTIVATION**

QCD thermodynamics on the lattice

Phase structure and $T_c$

Nature of the transition/crossover

$\mu=0$, $\mu \neq 0$

$\nu_s$, $\eta$, ....

> properties of quark matter at heavy ion collisions

> development of early universe / genesis of matter

Numerical simulations of QCD on the lattice

* the only systematic / ab initio way to investigate the issue

* becoming quantitatively powerful
**MOTIVATION**

status of lattice studies

\( T = 0 \quad N_F = 2+1 \) (degenerate ud + s) simulation

directly at the physical point \( (\text{PACS-CS Collab.}) \)

\[ \text{[PRD79, 034503]} \]

extrapolated to the phys. point

\[ \text{[arXiv: 0911.2561]} \]

directly at the phys. point by the reweighting method

\( 32^3 \times 64, a=0.09\text{fm} \)

planned: u-d mass difference and QED effects

yet to do: continuum extrapolation

\( T > 0, \mu \neq 0 \)

\( N_F = 2+1 \) at almost physical point

e.g. EOS with improved Staggered quarks:

RBC-Bi/MILC (‘07-‘09) \( N_f = 2 + 1, N_t = 4, 6, 8, m_\pi \sim 220, 150\text{MeV} \)

Budapest-Wuppertal (‘06) \( N_f = 2 + 1, N_t = 4, 6, m_\pi \sim 140\text{MeV} \)
MOTIVATION

lattice quarks

\[ T > 0, \mu \neq 0 \]

Most studies done with \textit{staggered-type quarks} because of less computational costs / a part of chiral sym. preserved / . . .

\[ N_f=2+1: \ T>0 \text{ at (almost) physical point, } \mu \neq 0 \text{ studies} \]

but

4 flavors degenerate ("taste")

\[ \leq \text{"4th root trick": replace det}D \text{ by its 4th root by hand} \]

\[ \Rightarrow \text{ non-local theory} \]

\[ \Rightarrow \text{ universality to our world not guaranteed} \]

Need crosschecks with theoretically sound lattice quarks.

* Wilson quarks much behind: no \( N_f=2+1 \), no \( \mu \neq 0 \), \( m_q \) not quite light . . .

* lattice chiral quarks (DW, overlap, . . .): yet in its infancy / too expensive

\textbf{We want to extend \textit{Wilson} studies} (thus “WHOT”-QCD)

\[ \text{to } N_f=2+1, \mu \neq 0, \text{ and smaller } m_q \]

Wilson simulations are more expensive \Rightarrow improvements/tricks are called for.
Motivation: Why Wilson-type quarks?

Improved Wilson quarks at \( \mu = 0 \)
with an introduction to \( T > 0 \) QCD on the lattice.

Improved Wilson quarks at \( \mu > 0 \)

Fixed scale approach with \( T \)-integral method

Summaries and outlooks
Wilson quarks at µ=0

PRD 63, 034502 (2001); PRD 64, 074510 (2001); PRD 75, 074501 (2007)
**INTRODUCTION**

$T>0$ QCD on the lattice

$$Z = \int [dU][dq][d\bar{q}] e^{-S}$$

$$T = \frac{1}{N_t a}$$

$N_t$ : lattice size in the euclidian time direction

$a$ : lattice spacing

Conventionally, $a$ is varied at a fixed $N_t$ ("fixed $N_t$ approach"), where $a$ is controlled by the lattice gauge coupling $\beta = \frac{1}{6g^2}$. 
THERMODYNAMICS WITH WILSON QUARKS

$N_f = 2$ QCD, $N_t = 4$, (6)

PRD 63, 034502 (2001); PRD 64, 074510 (2001);
PRD 75, 074501 (2007)

$$S = S_g + S_q,$$

$$S_g = -\beta \sum_x \left( c_0 \sum_{\mu<\nu;\mu,\nu=1}^4 W_{\mu\nu}^{1\times1}(x) + c_1 \sum_{\mu\neq\nu;\mu,\nu=1}^4 W_{\mu\nu}^{1\times2}(x) \right)$$

$$S_q = \sum_{f=1,2} \sum_{x,y} \bar{q}_x D_{x,y} q_y,$$

$$D_{x,y} = \delta_{xy} - K \sum_{\mu} \{(1 - \gamma_\mu)U_{x,\mu} \delta_{x+\mu,y} + (1 + \gamma_\mu)U_{x,\mu}^\dagger \delta_{x,y+\mu} \} - \delta_{xy} c_{SW} K \sum_{\mu<\nu} \sigma_{\mu\nu} F_{\mu\nu}$$

$c_1 = -0.331$, $c_0 = 1 - 8c_1$

$c_{SW} = (W^{1\times1})^{-3/4} = (1 - 0.8412\beta^{-1})^{-3/4}$

RG-improved Iwasaki glue
MF-improved clover quarks

$T>0$ simulations: $N_t = 4$, (6), $V = \text{mainly } 16^3$, $m_\pi \approx 500$ MeV
$T=0$: simulations: $16^4, 12^3 \times 24$
PHASE STRUCTURE

Chiral limit:
$m_\pi(T=0) = 0$

Light $q$:

Heavy $q$:

$k_c (T>0)$

parity-broken phase ($T=0$)

Chiral phase transition

Hot phase

Cold phase

$K_c (T=0)$

$K_l$

$N_t=4$

low $T$

large $a$

high $T$

small $a$

$T_c$ for $N_t = 4$
PHASE STRUCTURE

LCP’s (Lines of Constant Physics) determined by $m_\pi/m_\rho$

$m_\pi/m_\rho \approx 0.18$ in our world, $a \rightarrow 0$ to the left.

Finite $T$ crossover for $Nt=4$. [arXiv:0909.2121]
O(4) SCALING

- massless $N_f$-flavor QCD (continuum):
  \[ \text{SU}(N_f)_L \times \text{SU}(N_f)_R \times \text{U}(1)_V \times \text{U}(1)_A \]
  - SSB down to high T anomaly
  - \text{SU}(N_f)_V \times \text{U}(1)_V

  effective 3d $\sigma$ model (GL model)

  - $N_f \geq 3$: 1st order
  - $N_f = 2$: if anomaly negligible -> 1st order
    - with anomaly -> 2nd order $\iff$ O(4) Heisenberg model
  - O(4) scaling [crit. exponents, scaling functions, ...]

  (Note: RG flow enhances the anomaly towards the IR limit.)

\[ \frac{M}{h^{1/\delta}} = f\left(\frac{t}{h^{1/\beta \delta}}\right) \]

- reduced temperature $(T-T_c)/T_c$
- universal scaling function
- external magnetic field
- magnetization

\[ 1/\beta \delta = 0.537(7) \]
\[ 1/\delta = 0.2061(9) \]
O(4) SCALING

Fit QCD data with O(4) scaling function $f(x)$ and O(4) critical exponents.

\[ M/h^{1/\delta} = f(t/h^{1/\beta \delta}) \]

\[ t \sim \beta - \beta_{ct}, \ h \sim m_q a, \ \text{and} \ M \sim \langle \bar{\psi} \psi \rangle \]

\[ \langle \bar{\psi} \psi \rangle_{sub} = 2m_q a Z \sum_x \langle \pi(x) \pi(0) \rangle \]

To subtract out contributions from explicit chiral violation due to the Wilson term.

\[ \Rightarrow \] can be used to precisely extract $T_c$ in the chiral limit.

Current estimate: $T_c = 171 - 180 \ (N_t=4), \ 160 - 184 \ \text{MeV} \ (N_t=6)$ [WHOT-QCD, arXiv:0909.2121]

(cf.) Recent study with staggered quarks: Ejiri et al. arXiv:0909.5122
EOS

\[
\begin{align*}
\epsilon &= -\frac{1}{V} \frac{\partial \ln Z}{\partial T^{-1}}, \\
p &= T \frac{\partial \ln Z}{\partial V}
\end{align*}
\]

Trace anomaly:

\[
\frac{\epsilon - 3p}{T^4} = N_t^4 \left\{ \left\langle a \frac{dS}{da} \right\rangle - \left\langle a \frac{dS}{da} \right\rangle_{T=0} \right\} = \frac{N_t^3}{N_s^3} \sum_i a \frac{db_i}{da} \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\}
\]

\[
b = (\beta, \kappa_{ud}, \kappa_s, \cdots) \equiv (b_1, b_2, \cdots)
\]

lattice beta functions

scale-dep. of coupl. parameters along LCP

measured by the simulation.

T=0 subtracted for renormalization.
Beta functions not available for this combination of lattice actions yet. => We have to calculate by ourselves.

Inverse matrix method

i. Collect $T=0$ lattice results of #param. observables (e.g. $m_\pi a$, $m_\rho a$, ...).

ii. Fit them as functions of coupling parameters

iii. Determine LCP's

iv. Invert the coupling param. dependence of observables to the observable dependence of coupling parameters along a LCP.

\[
\begin{pmatrix}
\frac{\partial \beta}{\partial (m_V a)} & \frac{\partial K}{\partial (m_V a)} \\
\frac{\partial (m_V a)}{\partial K} & \frac{\partial (m_{PS}/m_V)}{\partial (m_{PS}/m_V)}
\end{pmatrix}^{-1} = \begin{pmatrix}
\frac{\partial (m_V a)}{\partial \beta} & \frac{\partial (m_{PS}/m_V)}{\partial \beta} \\
\frac{\partial (m_{PS}/m_V)}{\partial K} & \frac{\partial (m_{PS}/m_V)}{\partial K}
\end{pmatrix}
\]
EO5

Integral method:

Differentiate and integrate the thermodynamic relation

\[ p = \frac{T}{V} \int_{b_0}^{b} db \frac{1}{Z} \frac{\partial Z}{\partial b} = -\frac{T}{V} \int_{b_0}^{b} \sum_{i} db_i \left\{ \left\langle \frac{\partial S}{\partial b_i} \right\rangle - \left\langle \frac{\partial S}{\partial b_i} \right\rangle_{T=0} \right\} \]

such that \( p(b_0) \approx 0 \)

The integration path is free to choose as far as \( p(b_0) \approx 0 \)

[PRD63, 074510 (2001)]
Wilson quarks at $\mu \neq 0$
\[ \mu \neq 0 \text{ QCD on the lattice} \]

\[
U_4 \rightarrow U_4 e^{ia \mu} \quad \text{(positive direction)}; \quad U_4 e^{-ia \mu} \quad \text{(negative direction)}
\]

\[
Z = \text{Tr} e^{-H/T} = \int Dq D\bar{q} DU e^{-S}; \quad S = S_g + \sum \bar{q} M[U] q
\]

\[
M(\mu)^\dagger = \gamma_5 M(-\mu) \gamma_5
\]

\[
[\text{det} M(\mu)]^* = \text{det} M(-\mu) \neq \text{det} M(\mu)
\]

complex Boltzmann weight \[
\int Dq \, D\bar{q} e^{-\sum \bar{q} M[U] q} = \text{det} M(\mu)
\]

=> sign problem (complex phase problem)

Large cancellations due to phase fluctuations:

\[
\left\langle e^{i\theta} \right\rangle \propto e^{-V}
\]

Exponentially large statistics needed to keep the accuracy.
STATUS

Approaches to study at small \( \mu \):

- Reweighting from \( \mu=0 \)
- Taylor expansion from \( \mu=0 \)
- Analytic continuation from imag. \( \mu \)
- Canonical ensemble

Taylor expansion method with improved Staggered quarks:

- Bielefeld-Swansea ('02-05) \( \text{Nf}=2, \text{Nt}=4, \text{heavy} \)
- MILC, RBC-Bielefeld ('08-) \( \text{Nf}=2+1, \text{Nt}=4,6, \text{m}_\pi \approx 220\text{MeV} \)

No Wilson until ~'07.
STUDY WITH WILSON QUARKS AT $\mu \neq 0$

$N_f=2$ QCD, $N_t=4$ (based on the $\mu=0$ study) [arXiv:0909.2121]

- Observables

$$\frac{p}{T^4} = \frac{1}{V T^3} \ln \mathcal{Z} = \omega, \quad \frac{n_f}{T^3} = \frac{1}{V T^3} \frac{\partial \ln \mathcal{Z}}{\partial (\mu_f/T)} = \frac{\partial (p/T^4)}{\partial (\mu_f/T)}, \quad (f = u, d)$$

$$\chi_q \frac{T^2}{T^2} = \left( \frac{\partial}{\partial (\mu_u/T)} + \frac{\partial}{\partial (\mu_d/T)} \right) \frac{n_u + n_d}{T^3} \quad \chi_I \frac{T^2}{T^2} = \left( \frac{\partial}{\partial (\mu_u/T)} - \frac{\partial}{\partial (\mu_d/T)} \right) \frac{n_u - n_d}{T^3}$$

- Taylor expansion in $\mu_u=\mu_d(=\mu_q)$ at $\mu_u=\mu_d=0$

$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n, \quad c_n(T) = \frac{1}{n!} \frac{N_t^3}{N_s^3} \frac{\partial^n \ln \mathcal{Z}}{\partial (\mu_q/T)^n} \bigg|_{\mu_q=0}$$

$$\frac{\chi_q(T, \mu_q)}{T^2} = 2c_2 + 12c_4 \left( \frac{\mu_q}{T} \right)^2 + \cdots$$

$$c_2 = \frac{N_t}{2N_s^3} A_2, \quad c_4 = \frac{1}{4!N_s^3N_t} (A_4 - 3A_2^2)$$

$$A_2 = \langle D_2 \rangle + \langle D_1^2 \rangle, \quad A_4 = \langle D_4 \rangle + 4 \langle D_3D_1 \rangle + 3 \langle D_2^2 \rangle + 6 \langle D_2D_1^2 \rangle + \langle D_1^4 \rangle$$

$$\mathcal{D}_n = N_f \frac{\partial^n \ln \det M}{\partial \mu^n} \bigg|_{\mu = \mu_q a}$$

$$\mathcal{D}_1 = N_f \text{tr} \left( M^{-1} \frac{\partial M}{\partial \mu} \right)$$

$$\mathcal{D}_2 = N_f \left[ \text{tr} \left( M^{-1} \frac{\partial^2 M}{\partial \mu^2} \right) - \text{tr} \left( M^{-1} \frac{\partial M}{\partial \mu} M^{-1} \frac{\partial M}{\partial \mu} \right) \right]$$
NUMERICAL METHODS

Wilson quarks are more expensive => require improvements!

Noise method for traces

\[ \frac{1}{N_{\text{noise}}} \sum_i \eta_{i,\alpha} \eta^*_{i,\beta} \approx \delta_{\alpha,\beta} \]

\[ \text{tr} A = \sum_{\alpha} A_{\alpha,\alpha} \approx \frac{1}{N_{\text{noise}}} \sum_i \eta^\dagger_i A \eta_i \]

\[ \text{tr} \left( \frac{\partial^n M}{\partial^n \mu} M^{-1} \right) \approx \frac{1}{N_{\text{noise}}} \sum_i \eta^\dagger_i \frac{\partial^n M}{\partial^n \mu} X_i; \quad X_i = M^{-1} \eta_i \]

Noise method efficient when off-diagonal elements are small.

Because \( M^{-1} \) is a propagator, distant contributions are naturally suppressed.

For Wilson quarks, 3(color) x 4(spin) competitors at the same point.

=> We generate independent \( \eta \) for each color and spin.

(cf.) staggered quark: spins are staggered on a cube.

Error from \( D_1 \) turned out to be dominating in the results.

=> We adopt about 100 times larger \( N_{\text{noise}} \) for \( D_1 \).
RESULTS (1)

\( \mu = 0 \)

\( \mu = 0 \)

\( c_2 \)
Suggest critical pt. at finite \( \mu \), which is insensitive to the iso-spin number.
GAUSSIAN METHOD
To further improve the calculation

» A hybrid Taylor+reweighting method

Reweight the grand canonical partition function from \( \mu=0 \):

\[
\mathcal{Z}(T, \mu_q) = \mathcal{Z}(T, 0) \left\langle \left( \frac{\det M(\mu)}{\det M(0)} \right)^{N_f} \right\rangle_{(\mu_q=0)} \equiv \mathcal{Z}(T, 0) \left\langle e^{F(\mu)} e^{i \theta(\mu)} \right\rangle_{(\mu_q=0)}
\]

and Taylor-expand the terms in \( \exp \):

\[
F(\mu) \equiv N_f \text{Re} \left[ \ln \left( \frac{\det M(\mu)}{\det M(0)} \right) \right] \\
= N_f \sum_{n=1}^{\infty} \frac{1}{(2n)!} \text{Re} \left[ \frac{\partial^{2n}(\ln \det M)}{\partial \mu^{2n}} \right]_{(\mu=0)} \mu^{2n} = \sum_{n=1}^{\infty} \frac{1}{(2n)!} \text{Re} D_{2n} \mu^{2n}
\]

\[
\theta(\mu) = N_f \text{Im} \left[ \ln \det M(\mu) \right] \\
= N_f \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \text{Im} \left[ \frac{\partial^{2n+1}(\ln \det M(\mu))}{\partial \mu^{2n+1}} \right]_{(\mu=0)} \mu^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \text{Im} D_{2n+1} \mu^{2n+1}
\]

Truncate the expansions up to \( D_4 \)

* Identical to the truncated Taylor expansion up to the 4th order, but contain a part of higher orders through the exponential function.
* Exact for free QGP, in which \( D_n=0 \) for \( n>4 \), \( \Rightarrow \) The truncation will be OK at high \( T \).
Gaussian approximation for $\theta$

$B_4 = 3$ if Gaussian

$\Rightarrow \theta$-integration tractable by measuring

This mildens the sign problem (fluctuation due to $e^{i\theta}$).

(Note) $\theta$ as defined by the expansion is not restricted between $-\pi$ and $+\pi$. Conventionally periodic $\theta$ is recovered by mapping the result into $(-\pi, +\pi]$. [arXiv:0909.2121]
The quark number susceptibility for each $\mu_q/T$ at $m_{PS}/m_Y = 0.65$ (left) and 0.80 (right).
RESULTS (3)

heavy quark free energies

\[ V^R(r, T, \mu_q) = v_0^R + v_1^R \left( \frac{\mu_q}{T} \right) + v_2^R \left( \frac{\mu_q}{T} \right)^2 + O(\mu^3) \]

\( \bar{Q}Q \quad QQ \)

\( R = 1, 8, 6, 3^* \)

\[ \Omega^1(r) = \frac{1}{3} \text{tr} \Omega^1(x) \Omega(y), \]

\[ \Omega^8(r) = \frac{1}{8} \text{tr} \Omega^1(x) \text{tr} \Omega(y) - \frac{1}{24} \text{tr} \Omega^1(x) \Omega(y), \]

\[ \Omega^6(r) = \frac{1}{12} \text{tr} \Omega(x) \text{tr} \Omega(y) + \frac{1}{12} \text{tr} \Omega(x) \Omega(y), \]

\[ \Omega^{3^*}(r) = \frac{1}{6} \text{tr} \Omega(x) \text{tr} \Omega(y) - \frac{1}{6} \text{tr} \Omega(x) \Omega(y), \]

\( \bar{Q}Q \) interaction:
weaker at \( \mu > 0 \)

\( QQ \) interaction:
stronger at \( \mu > 0 \)

(leading order in \( \mu \))
SUMMARY

- Study of finite density QCD with Wilson-type quarks possible by improvements.

- So far, results are consistent with previous studies with staggered quarks.
  
  - suggest a critical point at finite $\mu$
    
    - sensitive to quark number,
    
    - but insensitive to iso-spin number

- Some new results for heavy quark free energies, etc.
Fixed scale approach with T-integral method

MOTIVATION

We want to extend Wilson studies (thus “WHOT”-QCD) to $N_F=2+1$, larger $N_t$, and smaller $m_q$ to avoid theoretical uncertainties of stag. quarks (universality, locality, etc.).

More improvements of the method are longed for.

A large fraction of the cost: $T=0$ simulations for both stag. and Wilson

- Determination of basic info about the simulation point: scale, non-perturbative beta function, etc.
- $T=0$ subtractions needed at all simulation points !!
- Determination of the Lines of Constant Physics (LCP)

To (partially) overcome the problem, we propose a fixed scale approach armed with a $T$-integral method.
FIXED SCALE APPROACH

Vary \( T = \frac{1}{N_t a} \) by varying \( N_t \) with all coupling params. fixed.

Conventional integral method not applicable.

\( T \)-integral method

A thermodynamic relation at \( \mu = 0 \):

\[
T \frac{\partial}{\partial T} \left( \frac{p}{T^4} \right) = \frac{\epsilon - 3p}{T^4}
\]

\[
\frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\epsilon - 3p}{T^5}
\]

such that \( p(T_0) \approx 0 \)

Note: \( T \) can be precisely determined up to the common overall scale \( a \).
FIXED SCALE APPROACH + T-INTEGRAL METHOD

Pros and cons:

- A common T=0 simulation enough for all T=0 subtractions.
- We can even borrow publicly available high statistic configurations on ILDG.
- Automatically on a LCP w/o fine tuning.
- Most of the T=0 simulation costs removable.
- The resolution in $T$ is limited because $N_t$ is discrete. $\Rightarrow$ No problem for EOS (see below)

At high T: $T > 2-3T_c$
- $N_t$ too small
- Keep the lattice volume large.

Around $T_c$:
- More costs due to larger $N_t$.
- Keep the lattice spacing small.

fixed $N_t$ approach powerful at high T coarse at low T complementary!

Note: Finite volume effects when $N_s/N_t$ is small (physical effects).
### TEST OF THE METHOD


#### Quenched QCD (w/o Dynamical Quarks)

<table>
<thead>
<tr>
<th>Set</th>
<th>$\beta$</th>
<th>$\xi$</th>
<th>$N_s$</th>
<th>$N_t$</th>
<th>$r_0/a_s$</th>
<th>$a_s (fm)$</th>
<th>$L (fm)$</th>
<th>$a (dg^{-2}/da)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i1</td>
<td>6.0</td>
<td>1</td>
<td>16</td>
<td>3–10</td>
<td>5.35(±0.2)</td>
<td>0.093</td>
<td>1.5</td>
<td>$-0.098172$</td>
</tr>
<tr>
<td>i2</td>
<td>6.0</td>
<td>1</td>
<td>24</td>
<td>3–10</td>
<td>5.35(±0.2)</td>
<td>0.093</td>
<td>2.2</td>
<td>$-0.098172$</td>
</tr>
<tr>
<td>i3</td>
<td>6.2</td>
<td>1</td>
<td>22</td>
<td>4–13</td>
<td>7.37(3)</td>
<td><strong>0.068</strong></td>
<td>1.5</td>
<td>$-0.112127$</td>
</tr>
<tr>
<td>a2</td>
<td>6.1</td>
<td>4</td>
<td>20</td>
<td>8–34</td>
<td>5.140(32)</td>
<td>0.097</td>
<td>1.9</td>
<td>$-0.10704$</td>
</tr>
</tbody>
</table>

- $T_c \approx 290$ MeV
- $N_t \approx 7–8$
- $N_t \approx 10$
- $N_t \approx 10$

borrowing the results of previous $T = 0$ simulations

$T \sim 200-700$ MeV
(240-1010 MeV for a2)

- Lattice cutoff effects $\leq i1$ vs. $i3$
- Spatial volume effects at **low** $T$ $\leq i1$ vs. $i2$
- Small $N_t$ artifacts at **high** $T$ $\leq i2$ vs. $a2$ / conventional method
- Error from interpolation of $T$ $\leq i2$ vs. $a2$
TEST(1): ISOTROPIC LATTICES

Trace anomaly

- i1 vs. i3
  ⇒ Lattice cutoff effs. negligible at all $T$.
  (i1 sufficiently fine.)

- i1 vs. i2
  ⇒ Spatial vol. effects near $T_c$.
  (Physical effects)
  For higher $T$, spatial size of 1.5fm sufficient.

- i2 vs. fixed Nt=8 result
  ⇒ Consistent for $T \leq 600$ MeV
TEST(1): ISOTROPIC LATTICES

Pressure by T-integration

- Sufficient points to interpolate.
- Trapezoidal interpolation leads to almost identical values.
  \[ \frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\epsilon - 3p}{T^5} \]

\[ \frac{3p}{T^4} \]

\( \beta=6.0, \xi=1, N_s=16, \text{(C)} \)
\( \beta=6.0, \xi=1, N_s=16, \text{(T)} \)
\( \beta=6.2, \xi=1, N_s=22, \text{(C)} \)
\( \beta=6.2, \xi=1, N_s=22, \text{(T)} \)

\( \Rightarrow \) Systematic errors due to the limited resolution in \( T \) are small.
TEST(2): ANISOTROPIC LATTICE

Another test by anisotropic lattice: $i^2$ vs. $a^2$

$$\xi = \frac{a_s}{a_t} = 4$$

$\Rightarrow$ about 4 times finer resolution in $T = \frac{1}{N_t a_t}$ at similar $a_s$.

- $a^2$ points on the $i^2$ interpolation line.
- Well consistent up to $\sim 600$ MeV.
  $\Rightarrow$ Systematic errors due to the limited resolution in $T$ are under control.
EOS well consistent with each other.
The fixed scale approach is capable to calculate EOS precisely.
SUMMARY

Fixed scale approach + T-integral method

- Vary $T$ by varying $N_t$ with all coupling parameters fixed.
- Evaluate the pressure non-perturbatively by
  \[
  \frac{p}{T^4} = \int_{T_0}^{T} dT \frac{\epsilon - 3p}{T^5}
  \]
- Can largely reduce the cost for $T=0$ simulations.
  - We can even borrow public configurations on ILDG.
- Automatically on a LCP.
- Our approach is complementary to the fixed $N_t$ approach, and has advantages around $T_c$.

Tests in quenched QCD: promising

- Lattice cutoff effects small.
- Errors from the limited resolution in $T$ controllable and small.
- Results $\approx$ consistent with previous large scale study at fixed $N_t=8$. 
**EOS IN 2+1 FLAVOR QCD**

*N_f=2+1 QCD* simulation

based on a *T=0* study by the CP-PACS+JLQCD Collaboration

- RG-improved Iwasaki glue + NP clover-improved Wilson quarks
- (2 fm$^3$) lattice, $a=0.07, 0.1, 0.12$ fm, exact PHMC algorithm for s, etc.
- configurations available on [ILDG](https://prd78.011502(08)

$$\beta = 2.05, \ a = 0.07 \text{fm}, \ m_\pi/m_\rho = 0.63, \ 28^3 \times 56, \ 6000\text{traj}.$$  

**T > 0 simulations**

$32^3 \times N_t, \ N_t = 4, 6, \cdots, 16$

Scales set by $r_0=0.5$ fm.

Polyakov loop history after thermalization (simulation on-going)
Heavy quark free energy at $T>T_c$

$$F^1(r, T) = -T \ln \langle \text{Tr} \Omega^+(x) \Omega(y) \rangle$$

in the Coulomb gauge

No vertical adjustment needed for $F^1$ in the fixed scale approach.

(cf.) $F^1 \rightarrow V(r)$ is used as an input to adjust the constant term of $F^1$ in the fixed $Nt$ approach.

We have thus proved the $T$-insensitivity of $F^1$ at short distances.

Temperature effects down to shorter distances at higher $T$. 

$F^1 \rightarrow V(r) = F^1(T=0)$ at short distances.

$2F_Q = -2T \ln \langle \text{Tr} \Omega \rangle$

2 x single quark free energy

at long distances.

i.e. deconfinement

$V(r) = -\frac{\alpha_0}{r} + \sigma r + V_0$
Debye screening mass at $T>T_c$

$$F^1(r,T) = -\frac{\alpha(T)}{r} e^{-m_D(T)r}$$

$N_f=0$: Umeda et al., PRD 79, 051501 (2009) fixed scale

$N_f=2$: Maezawa et al., PRD 75, 074501 (2007) fixed $N_t$
EOS (status report)

Beta functions not available for this lattice action yet.

=> We have to calculate by ourselves.

Inverse matrix method using the CP-PACS+JLQCD data for $m_{PS}/m_{V}(ll)$, $m_{PS}/m_{V}(ss)$

Gluon contribution to the trace anomaly.

(cf.) $N_{f}=2$ case ($N_{t}=4$):
peak height of the trace anomaly $\sim 13$

- gluon $\sim 45$,
- quark $\sim -32$

peak lower at larger $N_{t}$
PERSPECTIVES

Beta functions

More precise data needed.
Reweighting method to directly calculate beta functions at the simulation point.

$N_F = 2 + 1$ QCD just at the physical point
$T=0$ configurations by the PACS-CS Collab.
under accumulation.
$T>0$ simulations just at the reweighted point.

Finite density

We can combine our approach with the Taylor expansion method, to explore $\mu > 0$. 
thank you!