Initial conditions for high-energy nuclear collisions

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Outline

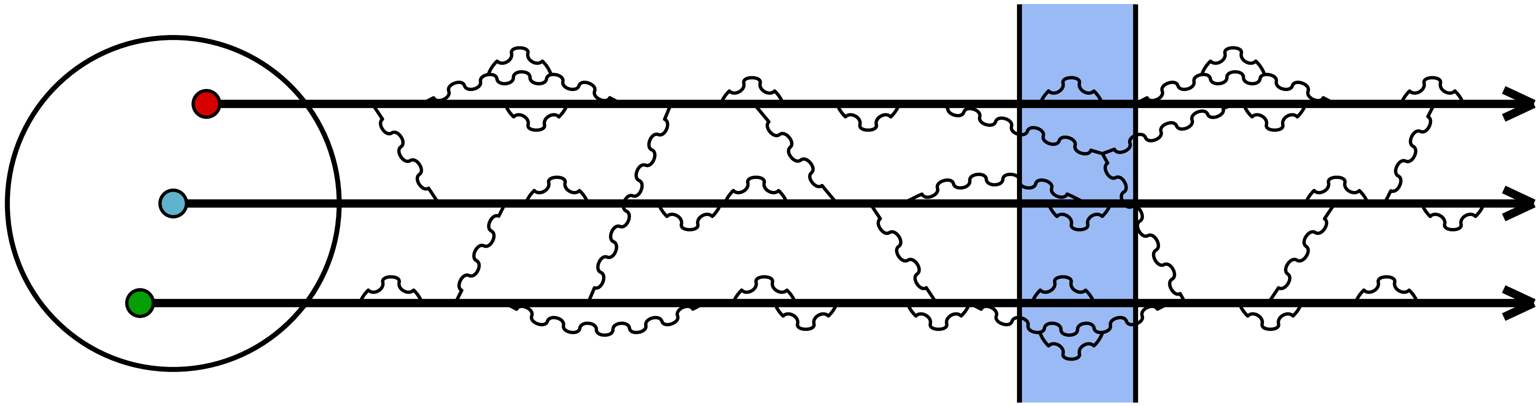
- Parton model, factorization, saturation
- Color Glass Condensate effective theory
- Power counting and bookkeeping
- Inclusive gluon spectrum at leading order
- Loop corrections, factorization

  - FG, Venugopalan, hep-ph/0601209, 0605246
  - Fukushima, FG, McLerran, hep-ph/0610416

  + work in preparation with Lappi, Venugopalan

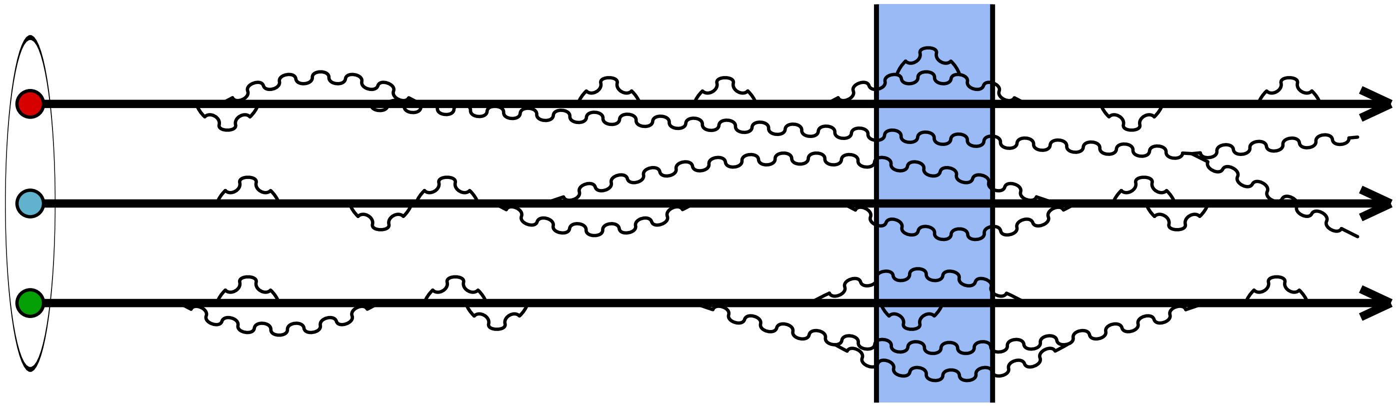
Parton model

Nucleon at rest



- A nucleon at rest is a very complicated object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe

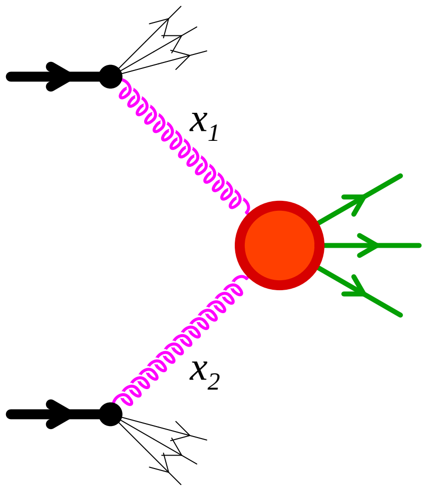
Nucleon at high energy



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe ▷ the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)
- Fast partons (fluctuations that were already visible before the boost) do not have any significant dynamics over the duration of the collision. They can be treated as static objects, that act as sources for the slower partons

Infrared and collinear divergences

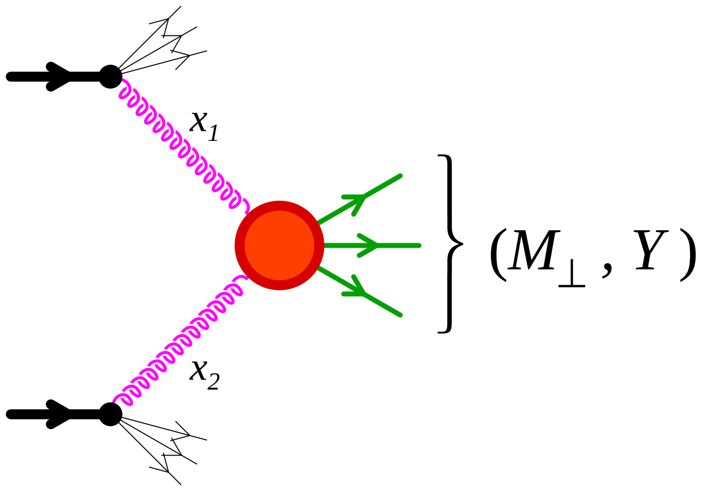
- Calculation of some process at LO :



$$\left.\right\} (M_\perp\,,\,Y) \qquad \begin{cases} x_1 = M_\perp\, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp\, e^{-Y}/\sqrt{s} \end{cases}$$

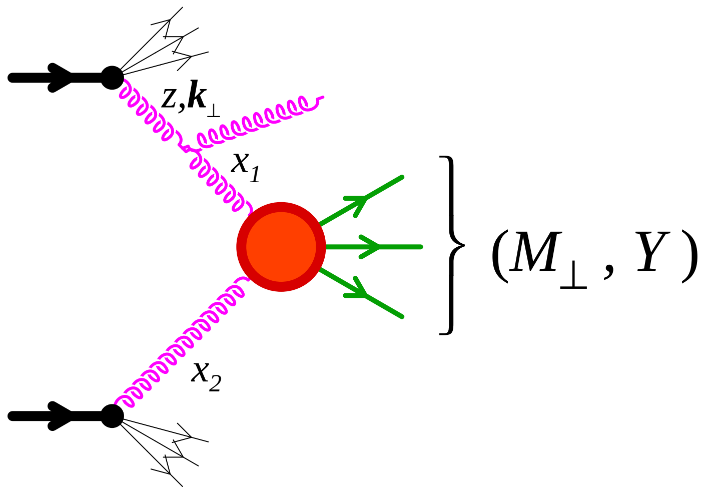
Infrared and collinear divergences

- Calculation of some process at LO :



$$\begin{cases} x_1 = M_\perp \, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp \, e^{-Y}/\sqrt{s} \end{cases}$$

- Radiation of an extra gluon :



$$\Longrightarrow \quad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

Infrared and collinear divergences

- Large logs : $\log(M_\perp)$ or $\log(1/x_1)$, under certain conditions

  ▷ these logs can compensate the additional α_s, and void the naive application of perturbation theory
  ▷ resummations are necessary

- Logs of $M_\perp \Longrightarrow$ DGLAP + Collinear factorization
  - $M_\perp \gg \Lambda_{QCD}$
  - x_1, x_2 are rather large

- Logs of $1/x \Longrightarrow$ BFKL + $k_\perp$-factorization
  - $M_\perp$ remains moderate
  - x_1 or x_2 (or both) are small

- Physical interpretation :
  - The physical process can resolve the gluon splitting if $M_\perp \gg k_\perp$
  - If $x_1 \ll 1$, the gluon that initiates the process is likely to result from bremsstrahlung from another parent gluon

Factorization in the linear regime

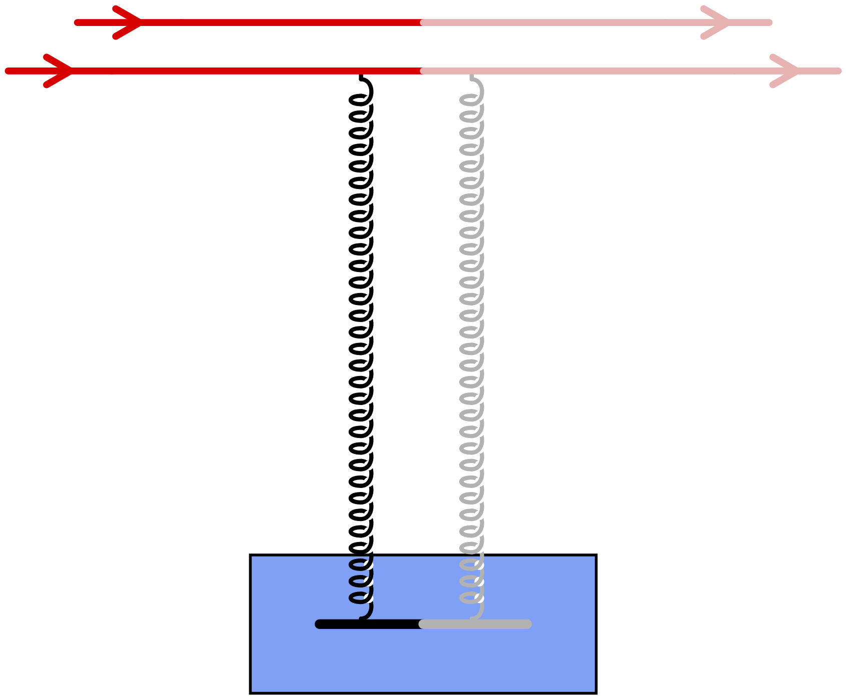
- Factorization in the linear small-x regime is known as k_{T}-factorization
- It was introduced in the discussion of heavy quark production near threshold, when $s \gg 4m_{\mathrm{q}}^2$, to resum large logs of $1/x_{1,2}$
  Collins, Ellis (1991), Catani, Ciafaloni, Hautmann (1991)
  Levin, Ryskin, Shabelski, Shuvaev (1991)
- In this framework, cross-sections read :

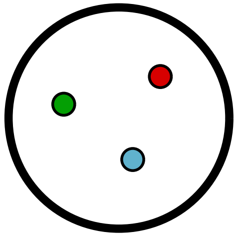
$$\frac{d\sigma}{dY\,d^2\vec{\boldsymbol{P}}_\perp} \propto \int_{\vec{\boldsymbol{k}}_{1\perp},\vec{\boldsymbol{k}}_{2\perp}} \delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}-\vec{\boldsymbol{P}}_\perp)\;\varphi_1(x_1,k_{1\perp})\;\varphi_2(x_2,k_{2\perp})\;\frac{|\mathcal{M}|^2}{k_{1\perp}^2\,k_{2\perp}^2}$$

$$x_{1,2} = \frac{M_\perp}{\sqrt{s}}\;e^{\pm Y}$$

- The small-x leading logs are resummed into the non-integrated gluon distributions $\varphi_{1,2}$ by letting them evolve according to the BFKL equation

Parton saturation



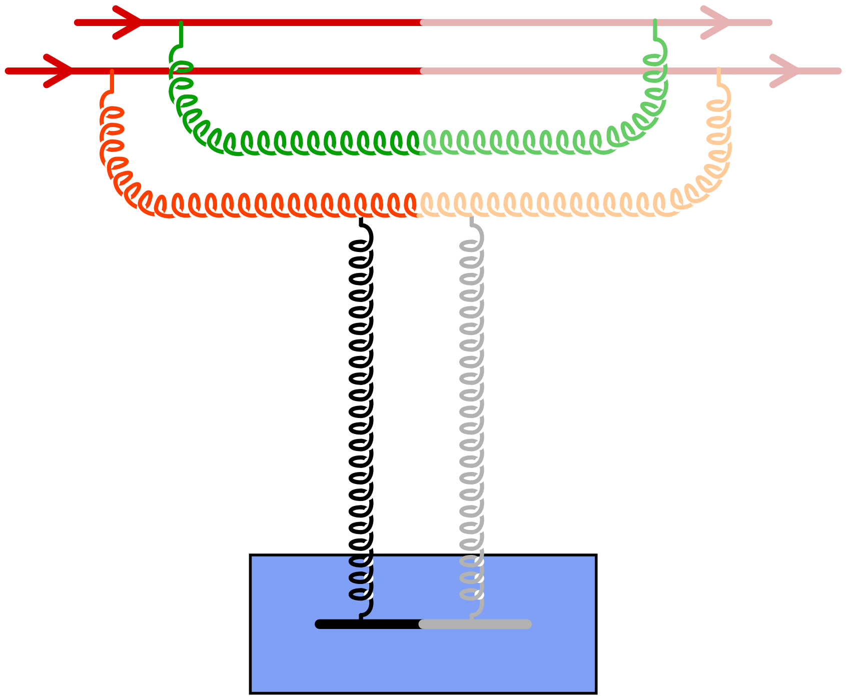


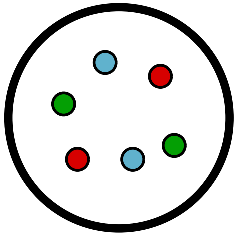
▷ assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)

▷ on the contrary, consider a small probe, with few partons

▷ at low energy, only valence quarks are present in the hadron wave function

Parton saturation



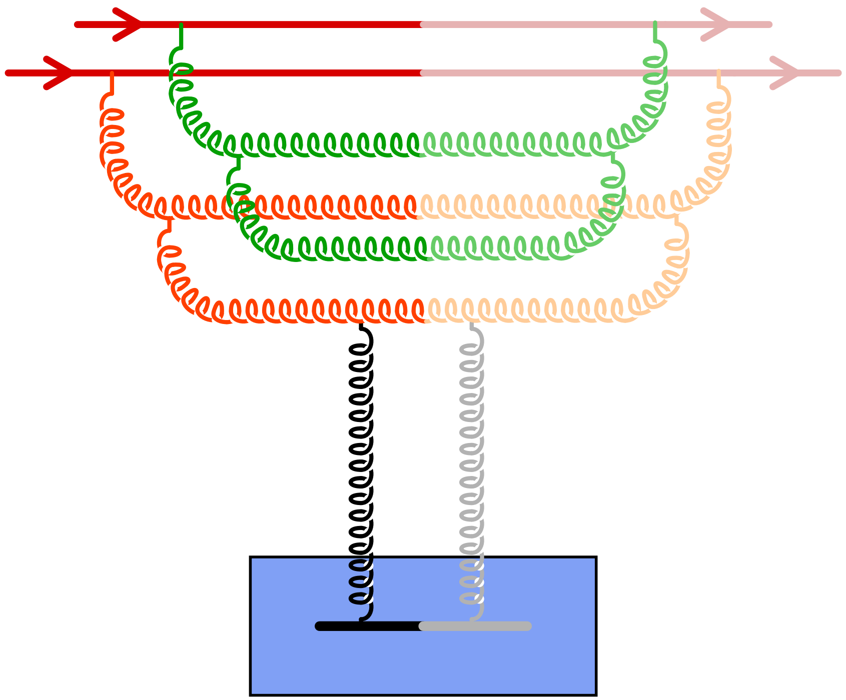


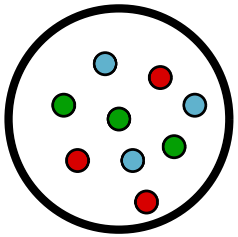
▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

▷ at small-x (i.e. high energy), these logs need to be resummed

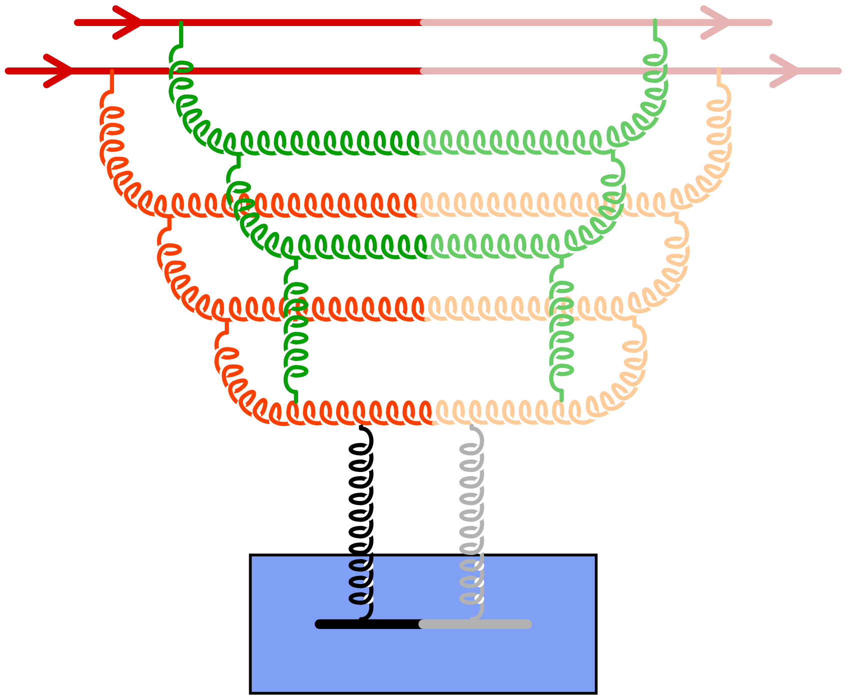
Parton saturation

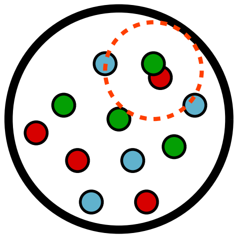




▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)

Parton saturation





▷ eventually, the partons start overlapping in phase-space

▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:
the number of partons created at a given step depends non-linearly on the number of partons present previously

Criterion for gluon recombination

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:

$$\rho \sim \frac{xG_A(x,Q^2)}{\pi R_A^2}$$

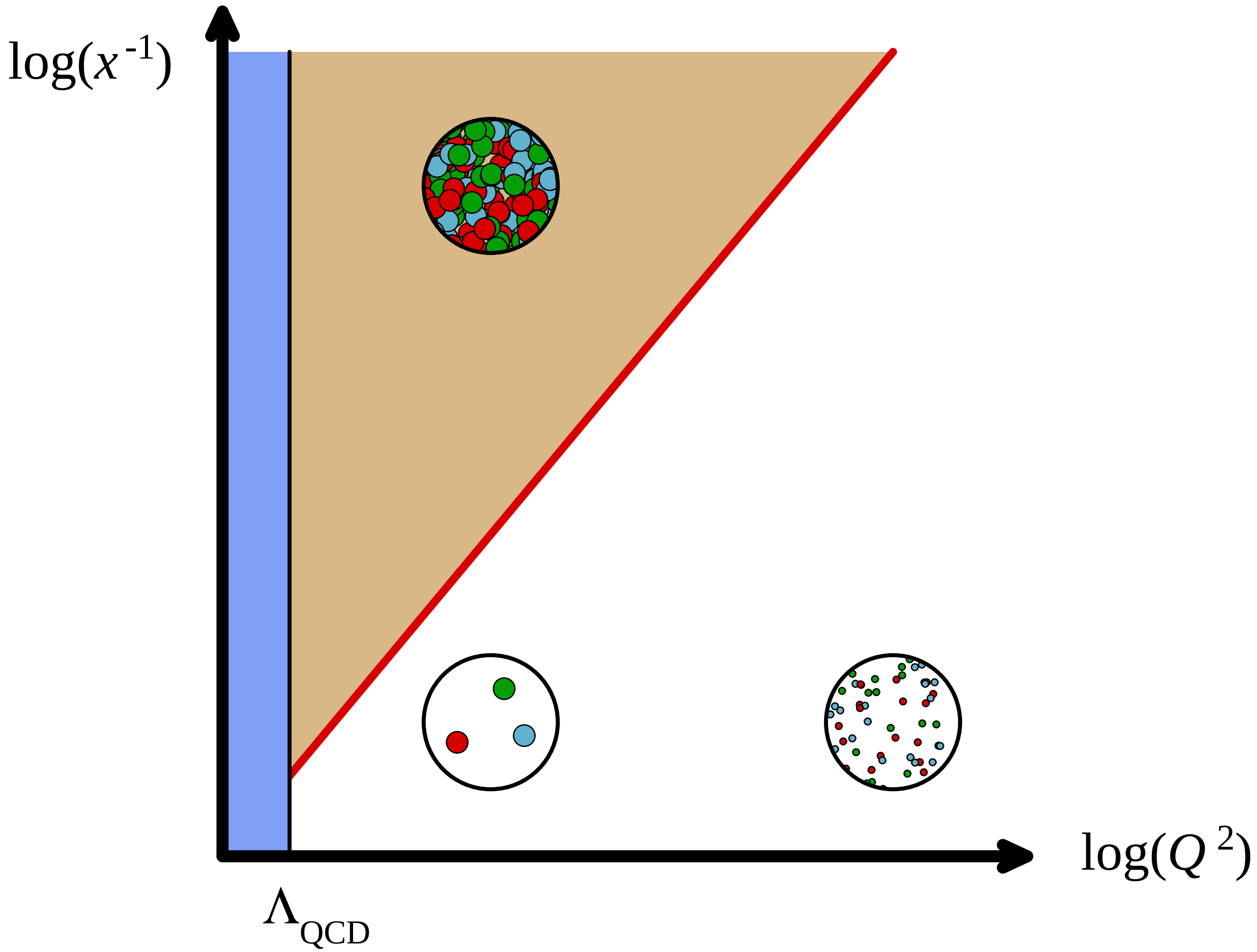
- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

- Recombination happens if $\rho\sigma_{gg\to g} \gtrsim 1$, i.e. $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \quad \sim \quad \frac{\alpha_s xG_A(x,Q_s^2)}{\pi R_A^2} \quad \sim \quad A^{1/3}\frac{1}{x^{0.3}}$$

Saturation domain



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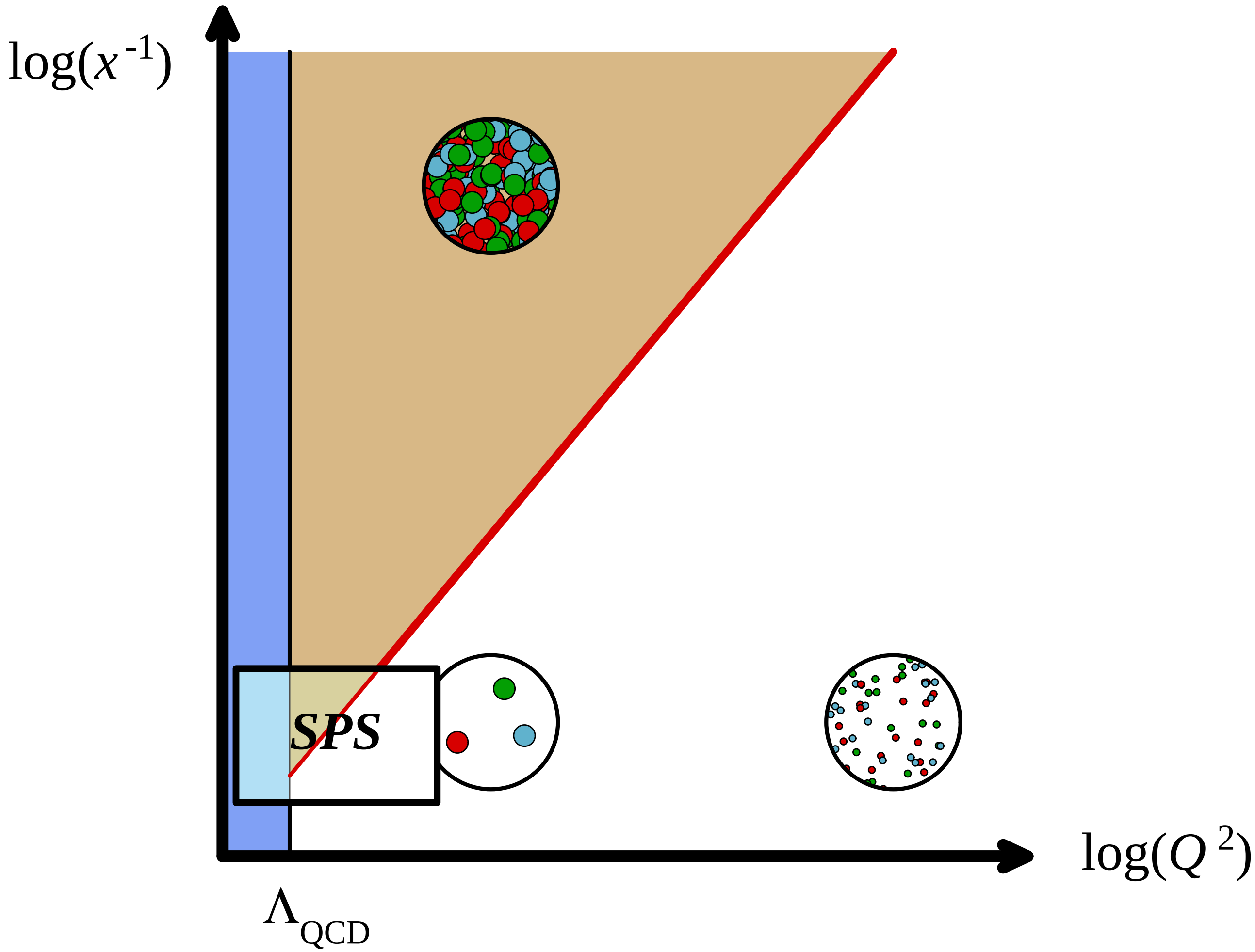

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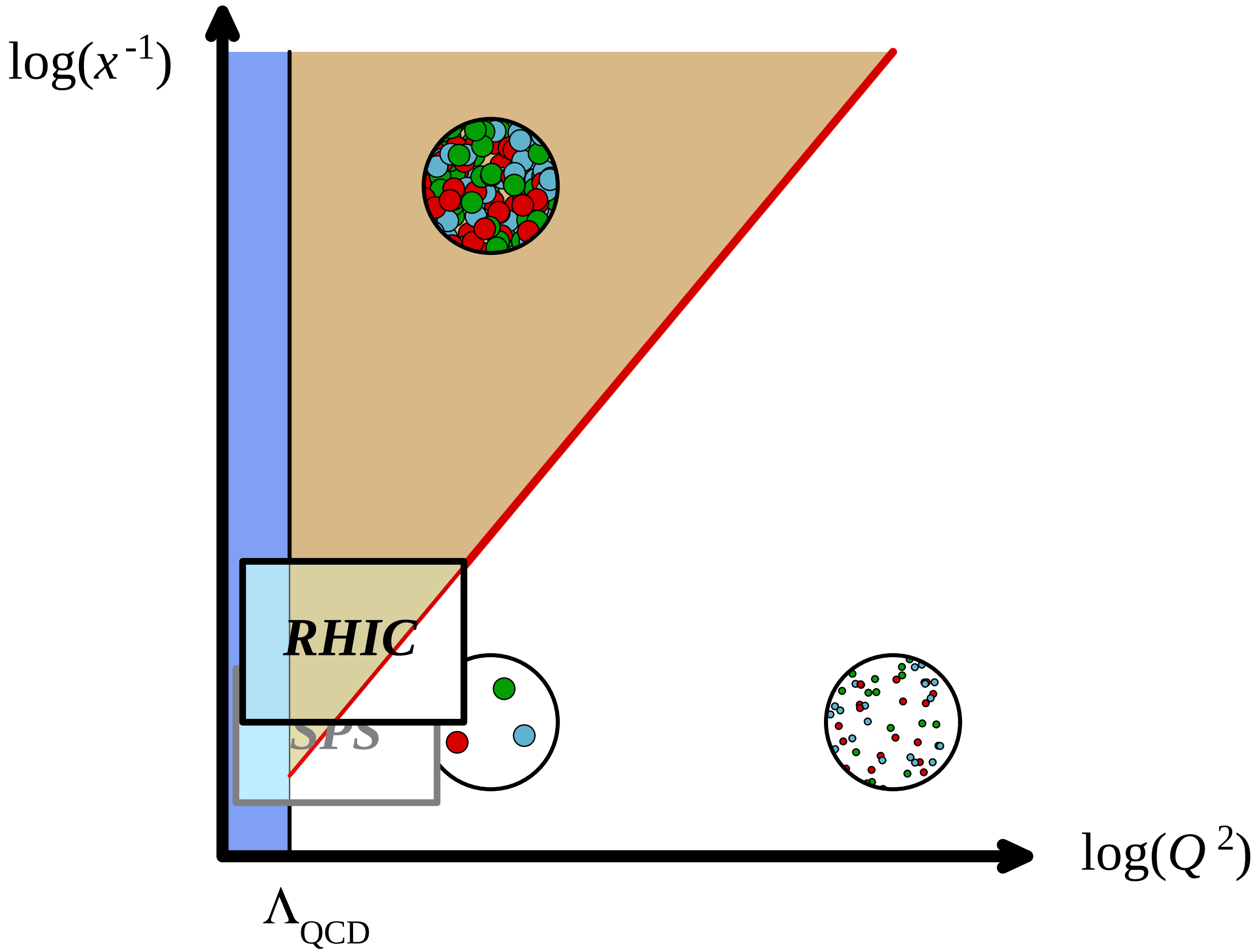

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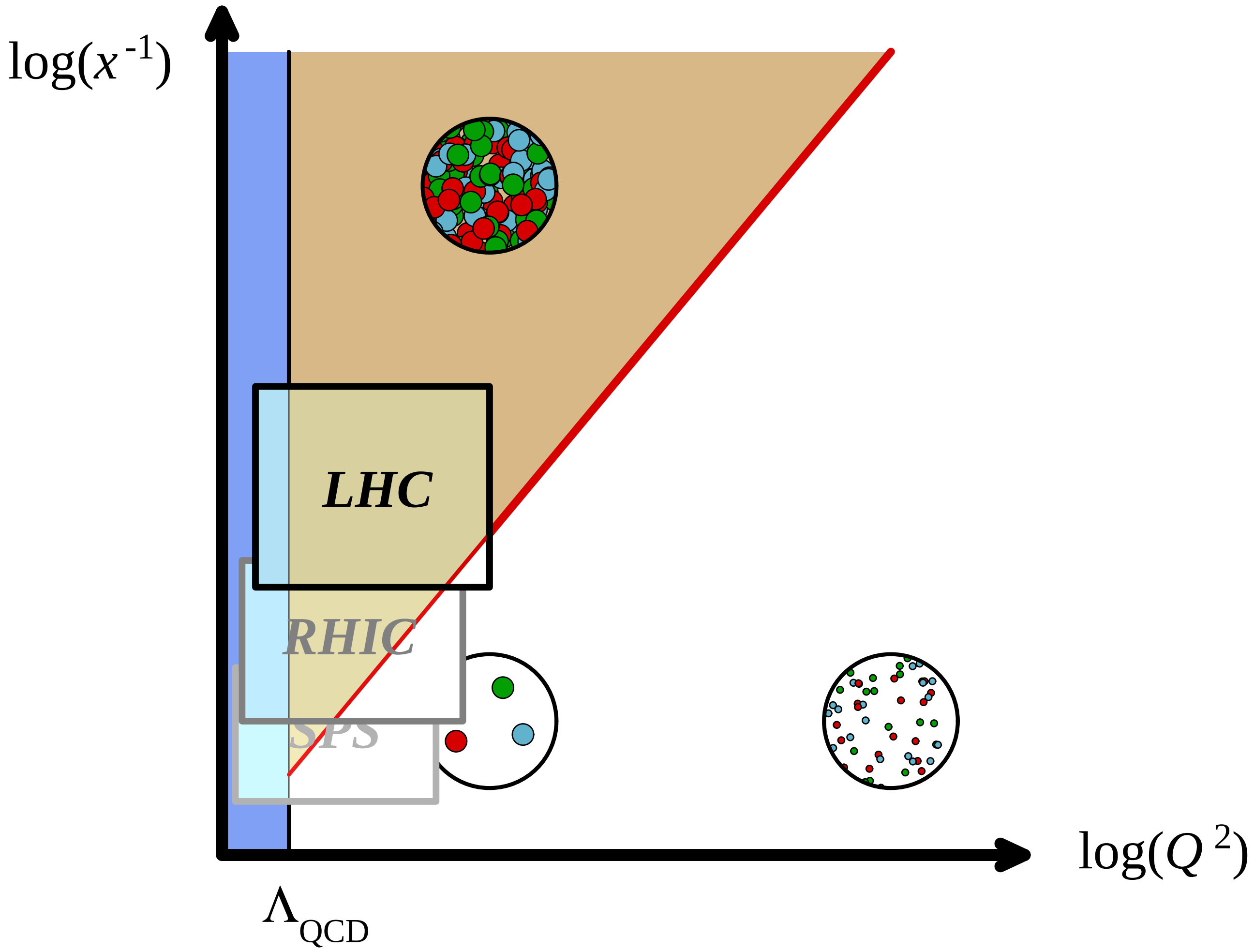

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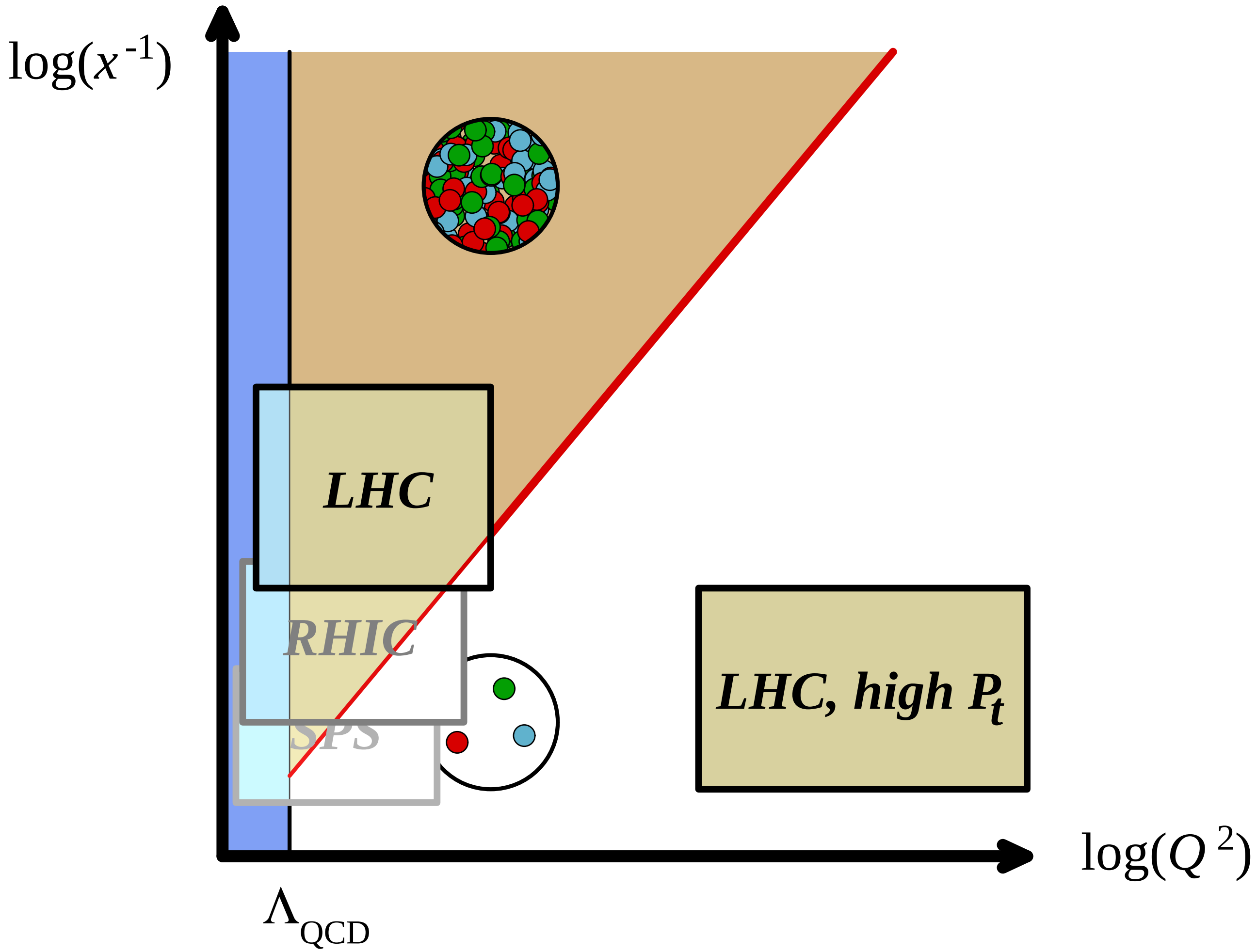


Multiple scatterings

- The saturation criterion can also be seen as a condition for multiple scatterings

- The mean free path of a gluon in a nucleus is

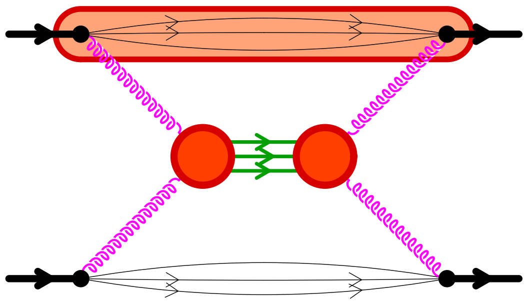
$$\lambda = \frac{1}{n\sigma_{gg\to g}} \quad , \quad n \sim \frac{xG_A(x, Q^2)}{\frac{4}{3}\pi R_A^3}$$

- Multiple scatterings are important if λ becomes smaller than the size of the nucleus, $\lambda \lesssim R_A$, i.e.

$$Q^2 \lesssim \alpha_s \frac{xG_A(x, Q^2)}{\pi R_A^2} \sim Q_s^2$$

Multiple scatterings

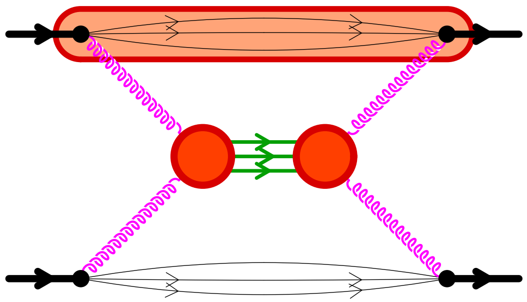
- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

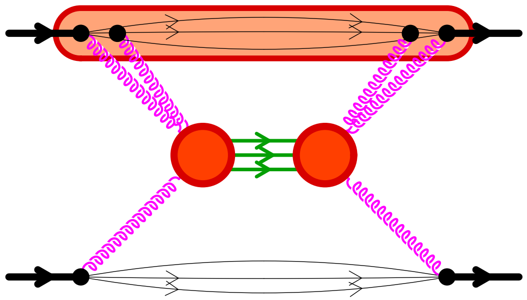
Multiple scatterings

- Single scattering :



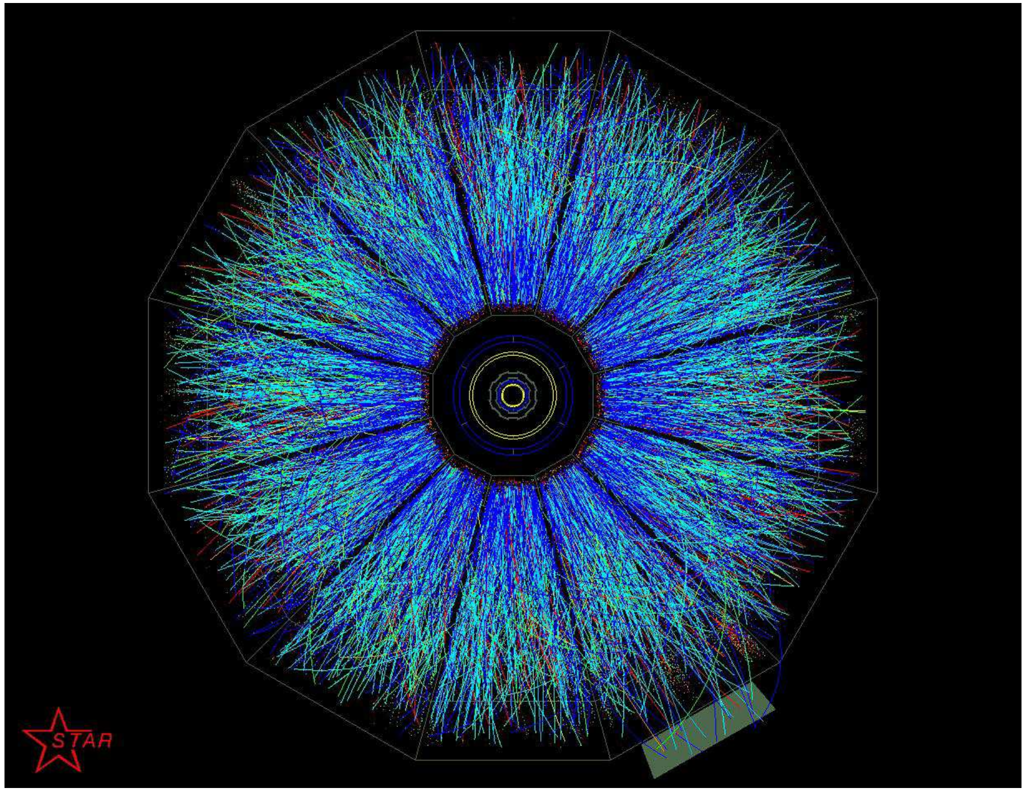
▷ 2-point function in the projectile ▷ gluon number

- Multiple scatterings :



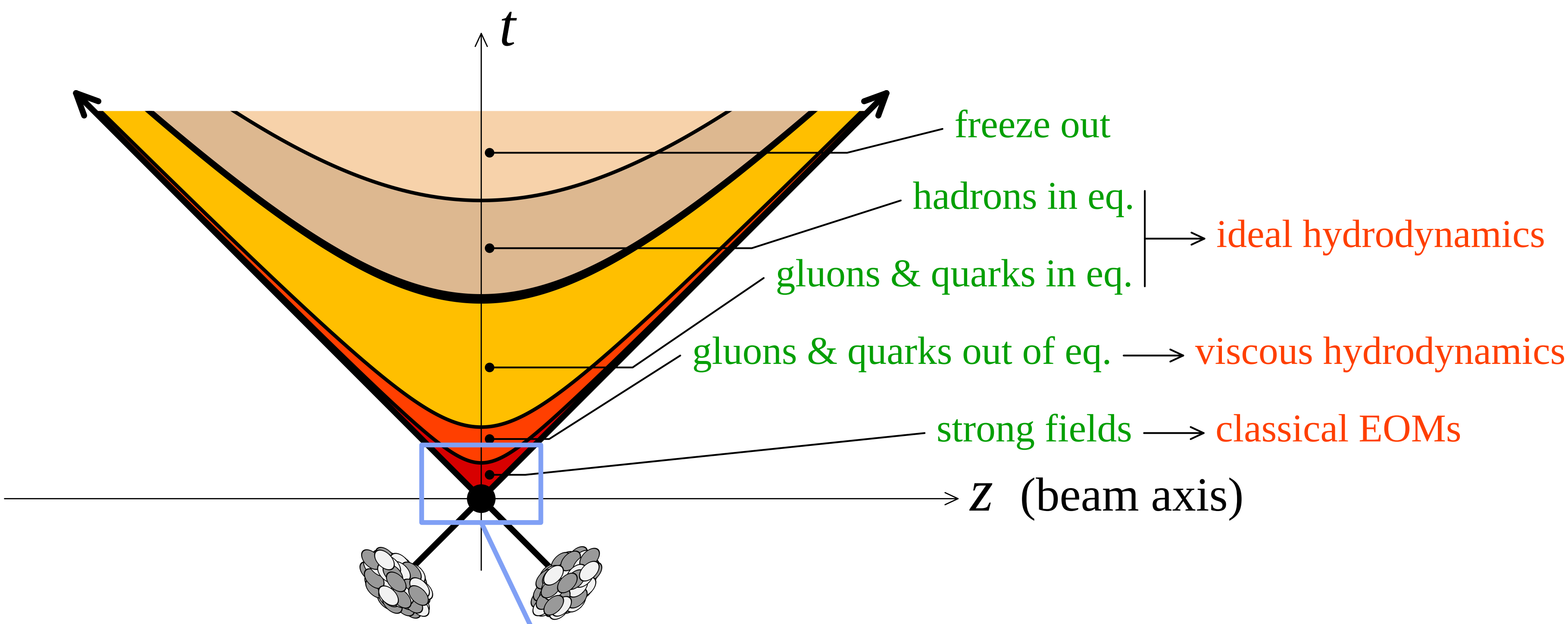
▷ 4-point function in the projectile ▷ higher correlations

Heavy Ion Collisions



- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- Q_s^2 might be as large as 10 GeV^2 at the LHC ($\sqrt{s} = 5.5$ TeV)
  - ▷ saturation and multiple scatterings will be important

Heavy Ion Collisions



- calculate the initial production of semi-hard particles
- provide initial conditions for hydrodynamics

Color Glass Condensate

Degrees of freedom

- The fast partons (large x) are frozen by time dilation
  $\triangleright$ described as static color sources on the light-cone :

$$J_a^\mu = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad\qquad (x^- \equiv (t-z)/\sqrt{2})$$

- Slow partons (small x) cannot be considered static over the time-scales of the collision process $\triangleright$ they must be treated as the usual gauge fields

  Since they are radiated by the fast partons, they must be coupled to the current J_a^μ by a term : $A_\mu J^\mu$

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates “soft” and “hard”

Color Glass Condensate

- Evolution equation (JIMWLK) :

$$\frac{\partial W_Y}{\partial Y} = \mathcal{H}\ W_Y$$

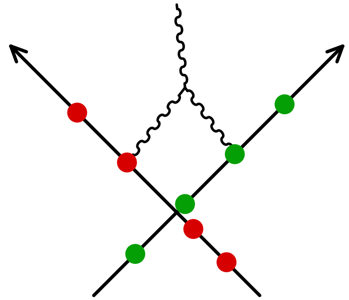
$$\mathcal{H} = \frac{1}{2}\int_{\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{\boldsymbol{y}}_\perp)}\eta_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{\boldsymbol{x}}_\perp)}$$

where $-\boldsymbol{\partial}_\perp^2\,\widetilde{\mathcal{A}}^+(\epsilon,\vec{\boldsymbol{x}}_\perp) = \rho(\epsilon,\vec{\boldsymbol{x}}_\perp)$

- η_{ab} is a non-linear functional of ρ
- This evolution equation resums the powers of $\alpha_s\ln(1/x)$ and of $Q_s/p_\perp$ that arise in loop corrections
- This equation simplifies into the BFKL equation when the source ρ is small (one can expand η_{ab} in ρ)

CGC and Nucleus-Nucleus collisions

- For symmetric collisions (e.g. nucleus-nucleus collisions), the two projectiles should be treated on the same footing
- For nucleus-nucleus collisions, there are two strong sources that contribute to the color current :



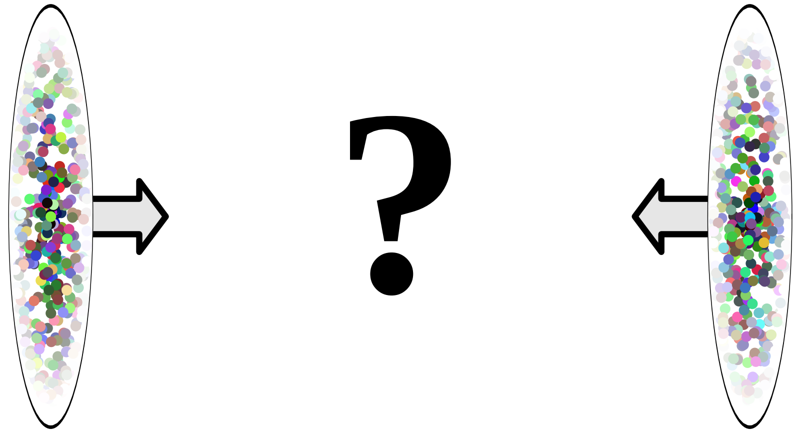
$$J^\mu \equiv \delta^{\mu +}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$

- Average over the sources ρ_1, ρ_2

$$\langle \mathcal{O}_{Y} \rangle_{\rho_1,\rho_2} = \int [D\rho_1]\,[D\rho_2]\; W_{Y_{\mathrm{beam}}-Y}\,[\rho_1]\; W_{Y+Y_{\mathrm{beam}}}\,[\rho_2]\; \mathcal{O}[\rho_1,\rho_2]$$

- How to compute $\mathcal{O}[\rho_1,\rho_2]$ in the saturation regime ?
- Can this factorization formula be justified ?

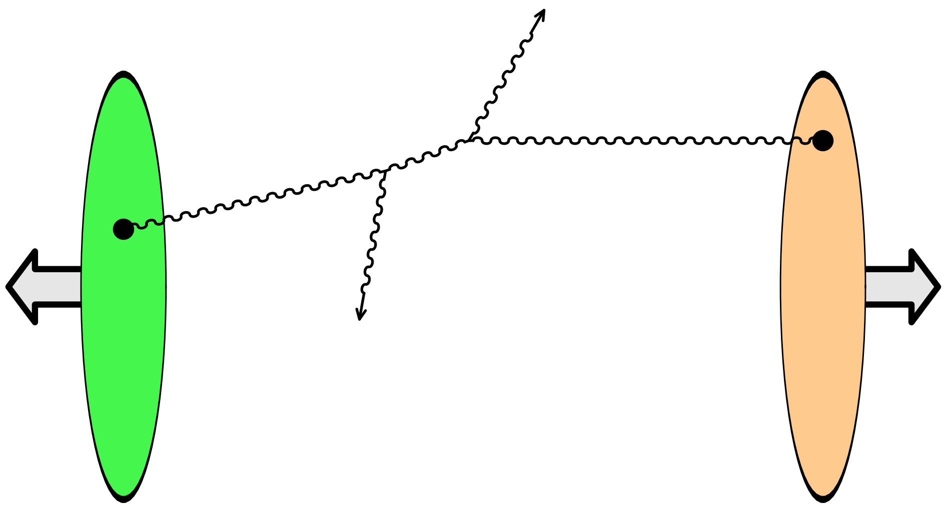
CGC and Nucleus-Nucleus collisions



$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu}A_\mu$$

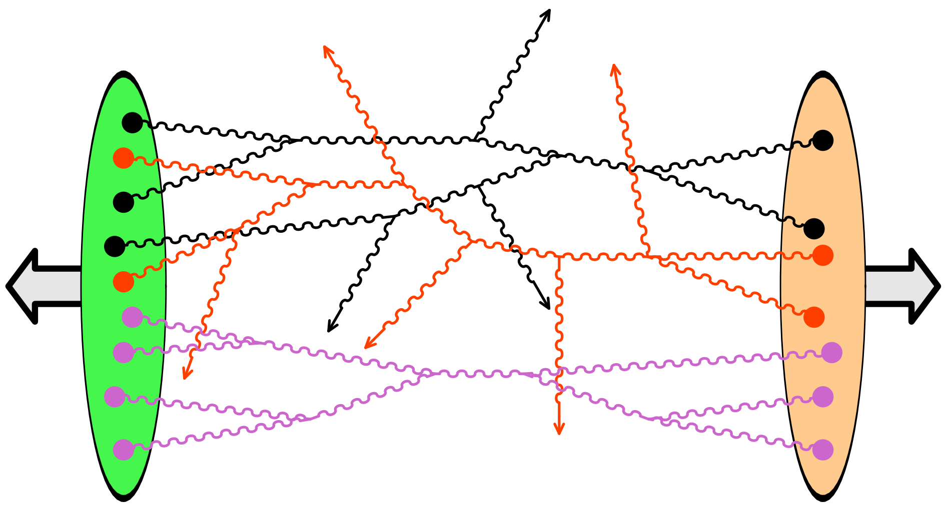
- Given the sources $\rho_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections and factorization?

Initial particle production



- Dilute regime : one parton in each projectile interact

Initial particle production



- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial

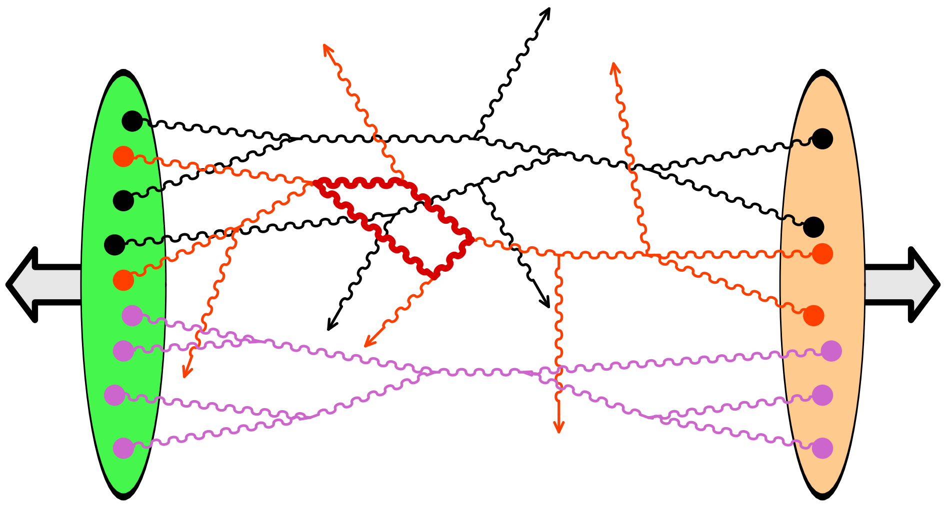
  (+ pileup of many partonic scatterings in each AA collision)

Goals

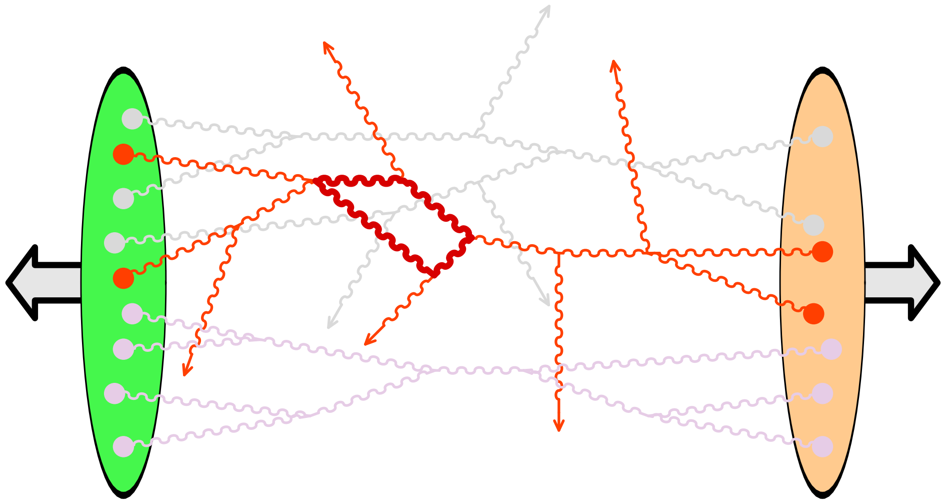
- Physics goal : provide as detailed as possible initial conditions for hydrodynamics. Ideally, one would like to have a CGC-based event generator for nucleus-nucleus collisions
- Factorization : can one resum the leading logs into objects that represent the initial state of the incoming nuclei ? Are these objects universal ?
- The easiest quantities to compute are event-averaged inclusive quantities. To have a true event generator, one should also predict particle correlations in a given event, and event by event fluctuations. Can we do this?

Power counting and Bookkeeping

Power counting



Power counting

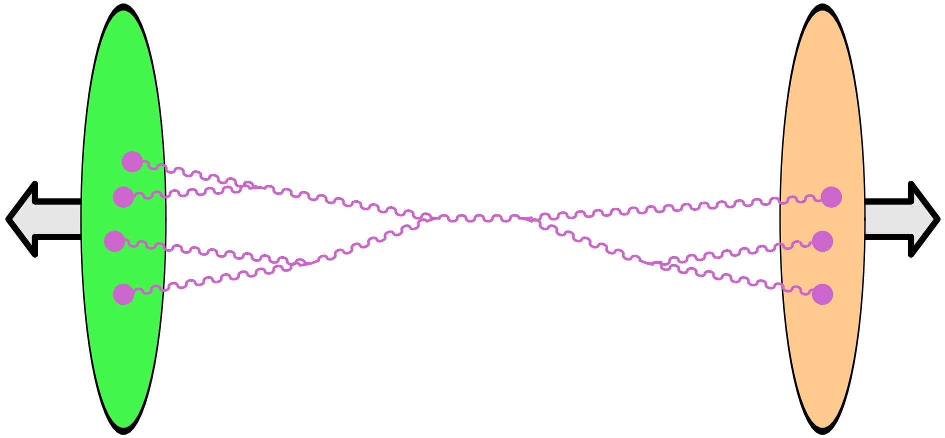


- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)
- The order of each connected diagram is given by :

$$\frac{1}{g^2}\quad g^{\#\ \text{produced gluons}}\quad g^{2(\#\ \text{loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams

Vacuum diagrams



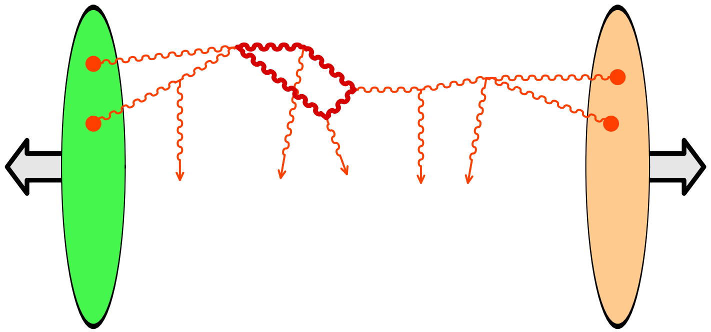
- Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\big\langle 0_{\rm out}\big|0_{\rm in}\big\rangle$
- The order of a connected vacuum diagram is given by :

$$g^{-2}\quad g^{2(\#\ \text{loops})}$$

- Relation between connected and non connected vacuum diagrams :

$$\sum\begin{pmatrix}\text{all the vacuum}\\ \text{diagrams}\end{pmatrix}=\exp\left\{\sum\begin{pmatrix}\text{simply connected}\\ \text{vacuum diagrams}\end{pmatrix}\right\}=e^{iV[j]}$$

Bookkeeping



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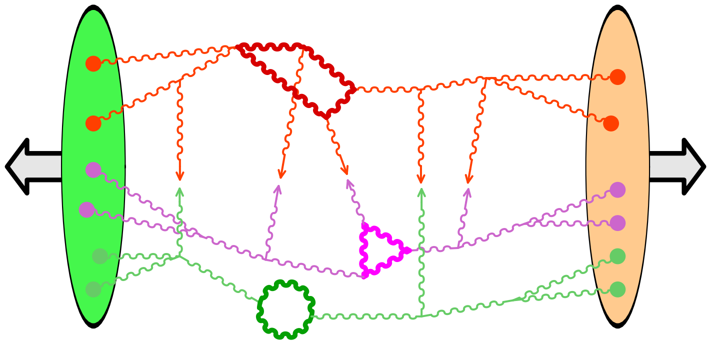
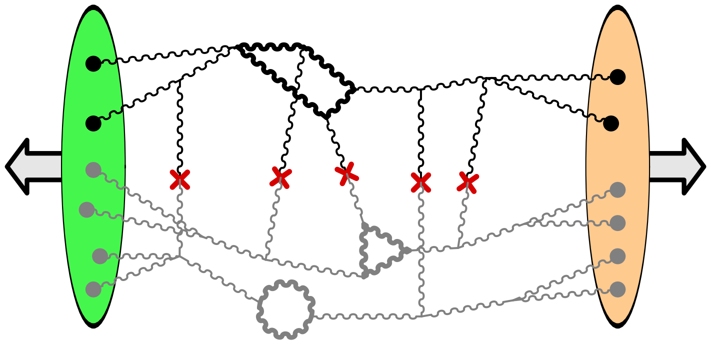


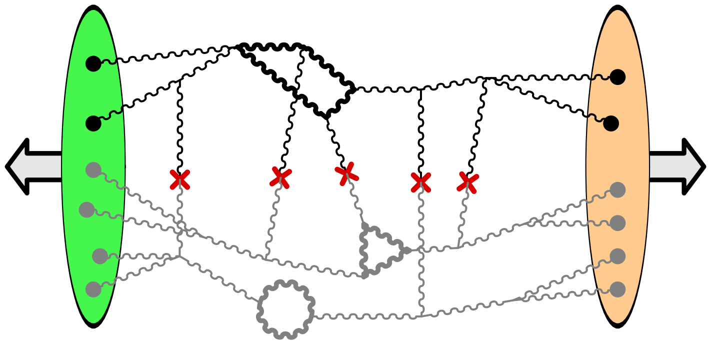
- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$
- The sum of the vacuum diagrams, $\exp(iV[j])$, is the generating functional for time-ordered products of fields :

$$\big\langle 0_{\rm out}\big|TA(x_1)\cdots A(x_n)\big|0_{\rm in}\big\rangle = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\; e^{iV[j]}$$

Bookkeeping

- The probability of producing exactly n particles is :

$$P_n = \frac{1}{n!}\int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\cdots\frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n}\left|\left\langle\vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle\right|^2$$

- There is an operator $\mathcal{C}$ such that
  - P_n is given by $P_n = \frac{1}{n!}\,\mathcal{C}^n\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$

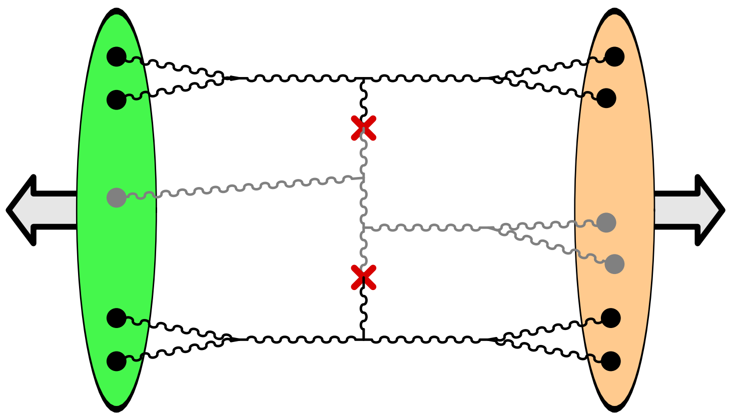
$$\text{with}\quad\begin{cases}\mathcal{C} \equiv \displaystyle\int_{x,y} G^0_{+-}(x,y)\,\square_x\square_y\,\frac{\delta}{\delta j_+(x)}\,\frac{\delta}{\delta j_-(y)}\\[2ex] G^0_{+-}(x,y) \equiv \displaystyle\int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(x-y)}\,2\pi\theta(-p^0)\delta(p^2)\end{cases}$$

Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :



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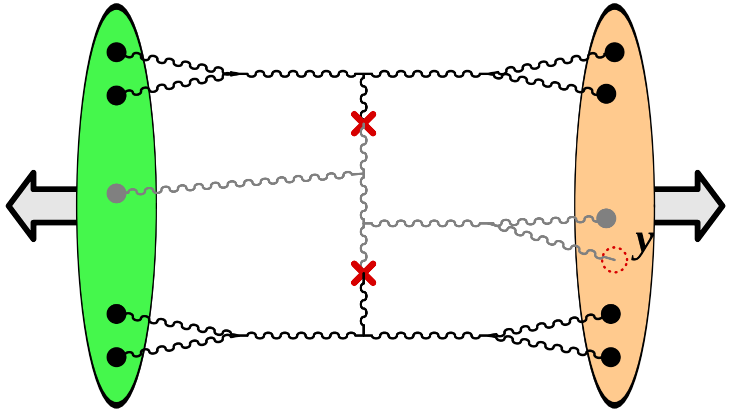

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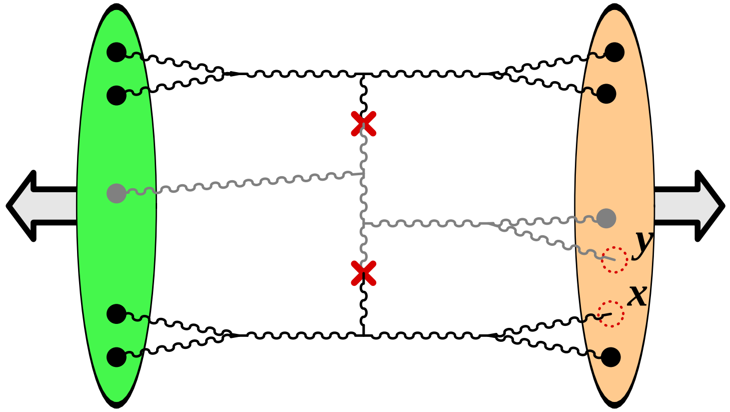

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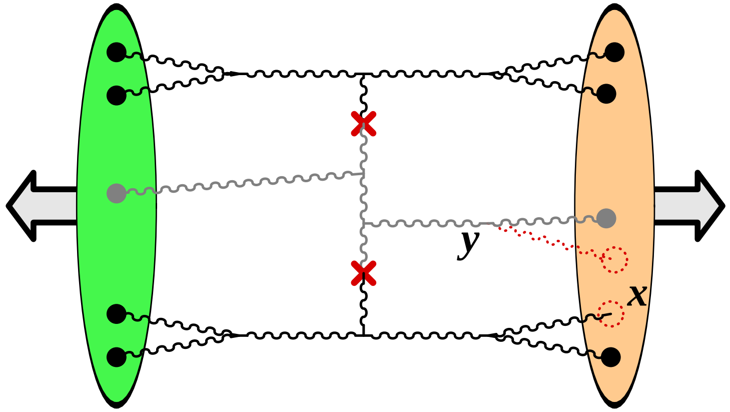
- Consider a generic cut vacuum diagram :

$$\square_y \quad \longrightarrow$$



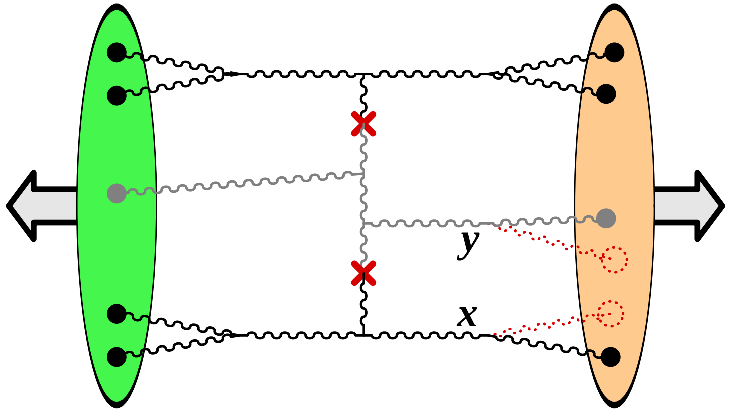
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x \square_y\,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\square_x \quad \longrightarrow$$



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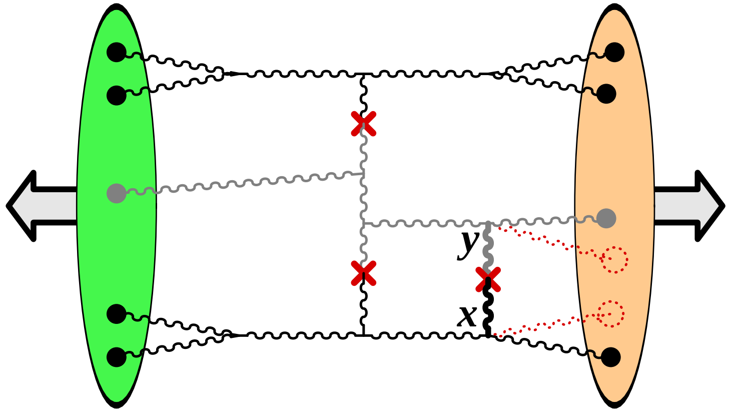


$\triangleright$ the operator $\mathcal{C}$ removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator

Bookkeeping

- The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$\sum \left(\begin{array}{c} \text{all the cut} \\ \text{vacuum diagrams} \end{array} \right) = e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}$$

- If we set $j_+ = j_- = j$, then we should get $\sum_n P_n = 1$

  (this property is a direct consequence of the “largest time equation” in Cutkosky’s cutting rules)

Bookkeeping

- The operator $\mathcal{C}$ can be used to derive many useful formulas :

$$F(z) \equiv \sum_{n=0}^{+\infty} z^n \, P_n = e^{z\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by z

$$\overline{N} = F'(1) = \mathcal{C} \, e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

$$\overline{N(N-1)} = F''(1) = \mathcal{C}^2 \, e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

- Benefits :
  - The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations
  - The use of the identity $e^{\mathcal{C}} \, e^{iV[j_+]} \, e^{-iV^*[j_-]}\Big|_{j_+=j_-} = 1$ renders automatic an important cancellation that would be hard to see at the level of diagrams (somewhat related to AGK)

Inclusive gluon spectrum

Diagrammatic expansion

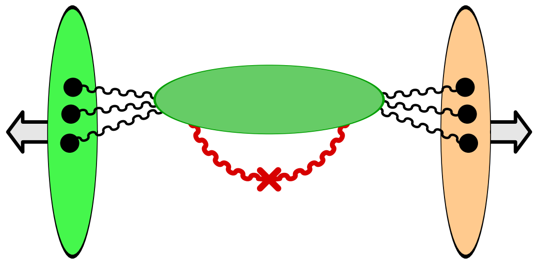
- It is easy to express the average multiplicity as :

$$\overline{N} = \sum_n n\, P_n = \mathcal{C}\left\{ \underbrace{e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}} \right\}_{j_+=j_-=j}$$

sum of all the cut vacuum diagrams : $e^{iW[j_+,j_-]}$

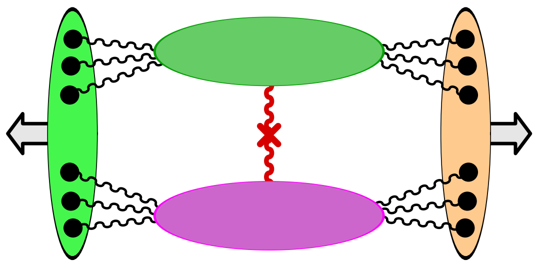
- There are two types of terms :
  - $\mathcal{C}$ picks two sources in the same connected cut diagram

$$\frac{\delta^2 iW}{\delta j_+(x)\delta j_-(y)} \quad \longrightarrow$$



  - $\mathcal{C}$ picks two sources in two distinct connected cut diagrams

$$\frac{\delta iW}{\delta j_+(x)}\;\frac{\delta iW}{\delta j_-(y)} \quad \longrightarrow$$

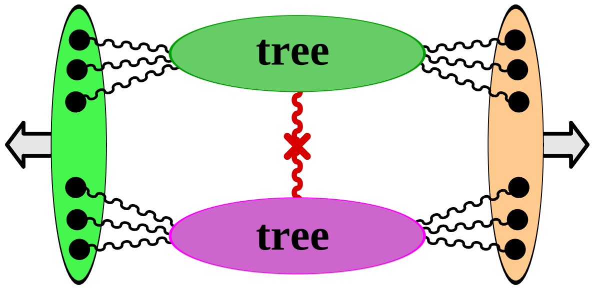


Single gluon spectrum at LO

- At LO, only tree diagrams contribute
  - ▷ the first type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- Reminder : at the end, the sources on both sides of the cut must be set equal :

$$j_+ = j_-$$

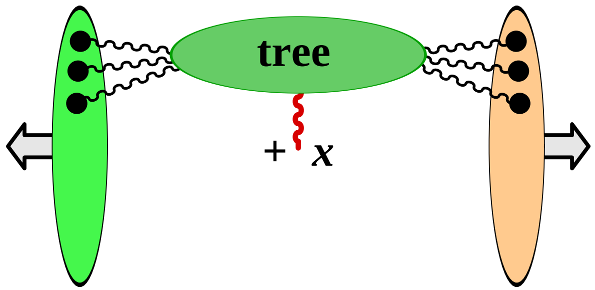
Expression in terms of classical fields

- The gluon spectrum at LO is given by :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y\sum_\lambda \epsilon_\mu^{(\lambda)}\epsilon_\nu^{(\lambda)}\;\mathcal{A}_+^\mu(x)\mathcal{A}_-^\nu(y)$$

- $\mathcal{A}_\pm^\mu(x)$ are sums of cut tree diagrams ending at the point x :

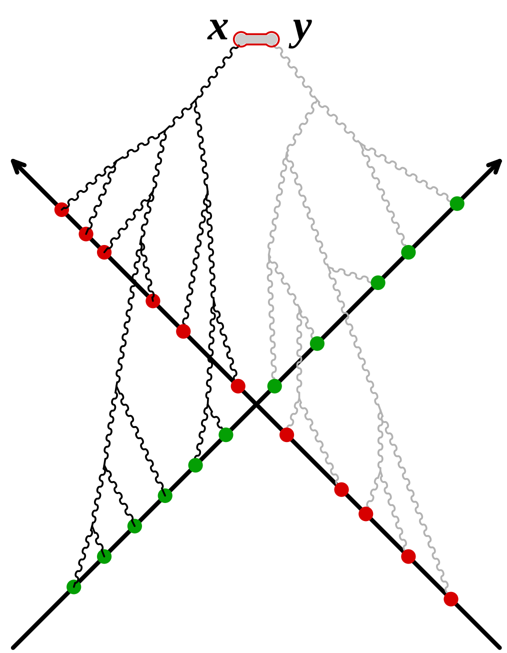
$$\mathcal{A}_+^\mu(x) = \sum_{\text{trees}}\sum_{\text{cuts}}$$



- $\mathcal{A}_\pm$ is a solution of the classical EOM ($\square\mathcal{A}_\pm + U'(\mathcal{A}_\pm) = j$)
- The boundary conditions are retarded : $\lim\limits_{x^0\to-\infty}\mathcal{A}_\pm(x) = 0$
- Because they have the same initial condition and are driven by the same source, $\mathcal{A}_+$ and $\mathcal{A}_-$ are equal

Single gluon spectrum at LO

- Retarded classical fields are sums of tree diagrams :



Single gluon spectrum at LO

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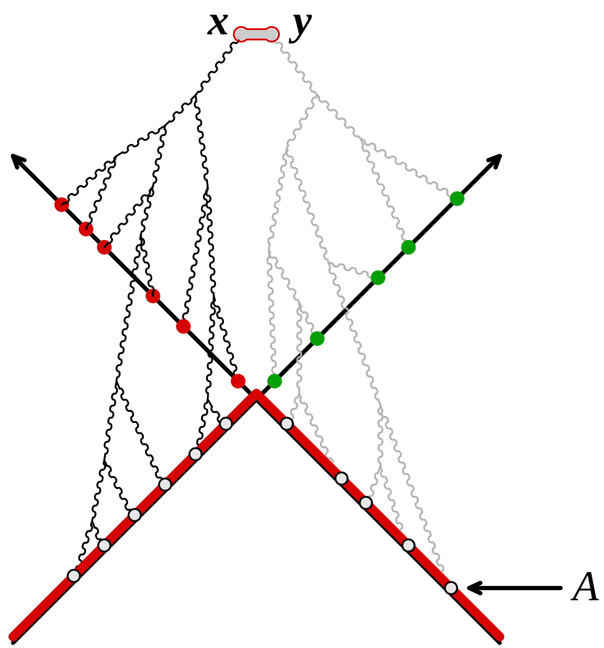
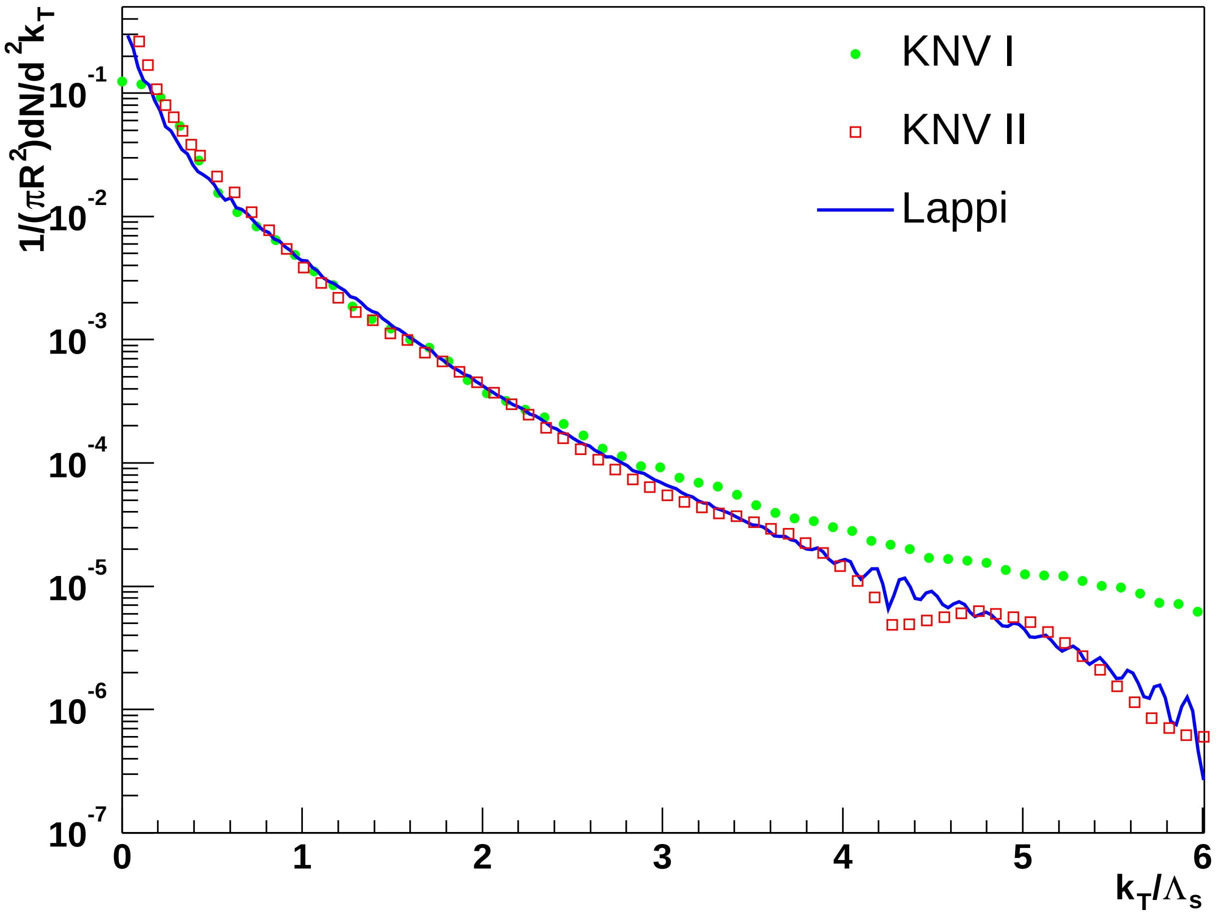


- Note : the gluon spectrum can be seen as a functional of the value of the classical field just above the backward light-cone :

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \mathcal{F}[\mathcal{A}]$$

Single gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)



- Important softening at small $k_\perp$ compared to pQCD (saturation)

Loop corrections, Factorization

Factorization in four easy steps

- I : Express the single gluon spectrum at LO and NLO in terms of classical fields and small field fluctuations. Check that their boundary conditions are retarded

- II : Write the NLO terms as a perturbation of the initial value of the classical fields on the light-cone :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

- III : For $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}$ on the same branch of the light-cone, one has :

$$\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}} = \log\left(\frac{\Lambda^+}{p^+}\right)\times\mathcal{H} + \text{finite terms}$$

- IV : These are the only logs. Factorization follows trivially

Single gluon spectrum at LO

- LO results for the single gluon spectrum :
  - Disconnected graphs cancel in the inclusive spectrum
  - At LO, the single gluon spectrum can be expressed in terms of classical solutions of the field equation of motion
  - These classical fields obey retarded boundary conditions

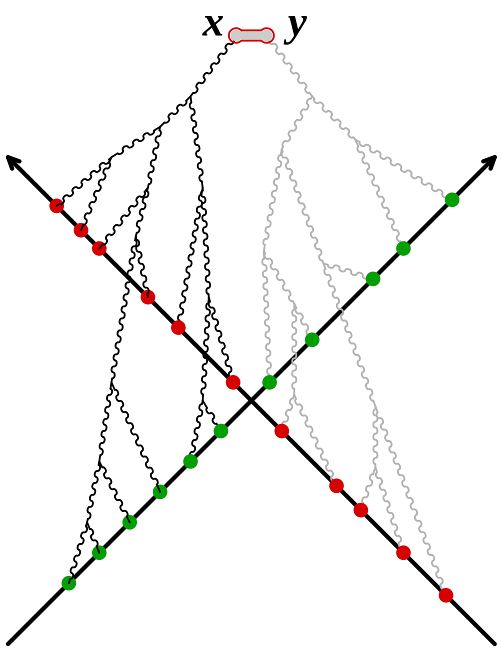
$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}} = \lim_{t\to+\infty} \int d^3\vec{\boldsymbol{x}} d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} \;\cdots \mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\;\mathcal{A}^\nu(t,\vec{\boldsymbol{y}})$$

$$\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = J^\nu$$

$$\lim_{t\to-\infty} \mathcal{A}^\mu(t,\vec{\boldsymbol{x}}) = 0$$

Single gluon spectrum at LO

- Retarded classical fields are sums of tree diagrams :



Single gluon spectrum at LO

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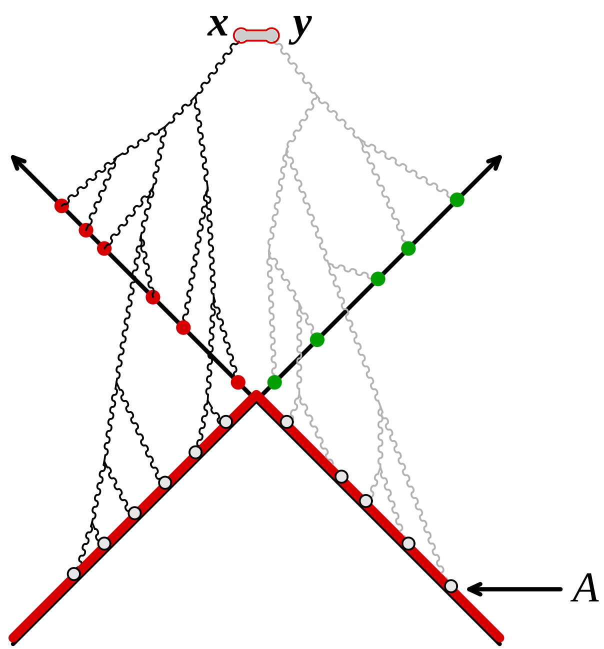
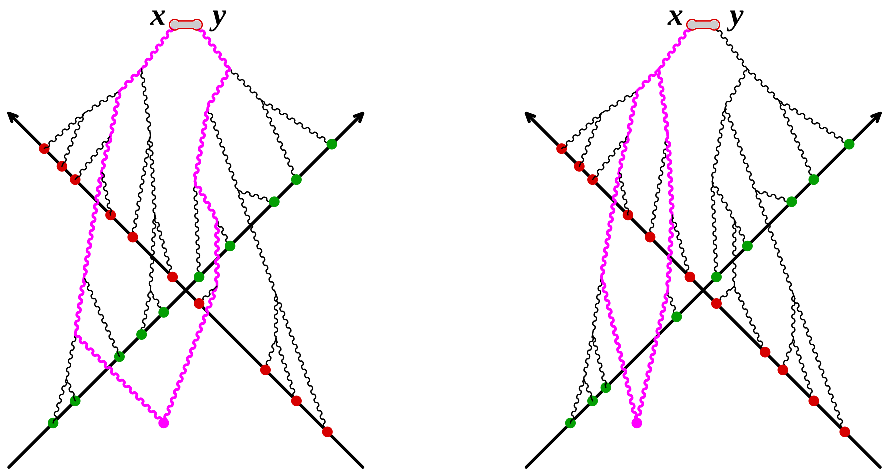


- Note : the gluon spectrum is a functional of the value of the classical field just above the backward light-cone :

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \mathcal{F}[\mathcal{A}]$$

Single gluon spectrum at NLO

- 1-loop graphs contributing to the gluon spectrum at NLO :



$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \lim_{t\to+\infty} \int d^3\vec{\boldsymbol{x}} d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})} \;\cdots \Big[\mathcal{G}^{\mu\nu}_{-+}(x,y) + \beta^\mu(t,\vec{\boldsymbol{x}})\,\mathcal{A}^\nu(t,\vec{\boldsymbol{y}}) + \mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\,\beta^\nu(t,\vec{\boldsymbol{y}})\Big]$$

- $\mathcal{G}^{\mu\nu}_{-+}$ is a 2-point function
- β^μ is a small field fluctuation driven by a 1-loop source

Single gluon spectrum at NLO

- The equation of motion for β^μ reads

$$\left[\mathcal{D}_\mu,\left[\mathcal{D}^\mu,\beta^\nu\right]\right]-\left[\mathcal{D}_\mu,\left[\mathcal{D}^\nu,\beta^\mu\right]\right]+ig\left[\mathcal{F}_\mu{}^\nu,\beta^\mu\right]=$$
$$=\frac{1}{2}\underbrace{\frac{\partial^3\mathcal{L}_{YM}(\mathcal{A})}{\partial\mathcal{A}^\nu(x)\partial\mathcal{A}^\rho(x)\partial\mathcal{A}^\sigma(x)}}_{\text{3-gluon vertex in the background }\mathcal{A}}\;\mathcal{G}^{\rho\sigma}_{++}(x,x)$$

- The 2-point functions $\mathcal{G}^{\mu\nu}_{-+}$ and $\mathcal{G}^{\mu\nu}_{++}$ can be written as

$$\mathcal{G}^{\mu\nu}_{-+}(x,y)=\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\eta^\mu_{-\boldsymbol{k}}(x)\;\eta^\nu_{+\boldsymbol{k}}(y)$$

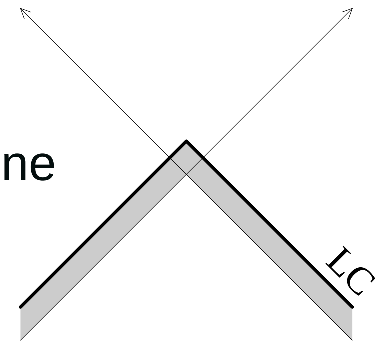
$$\mathcal{G}^{\mu\nu}_{++}(x,x)=\frac{1}{2}\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\left[\eta^\mu_{-\boldsymbol{k}}(x)\;\eta^\nu_{+\boldsymbol{k}}(x)+\eta^\mu_{+\boldsymbol{k}}(x)\;\eta^\nu_{-\boldsymbol{k}}(x)\right]$$

with
$$\begin{cases}\left[\mathcal{D}_\mu,\left[\mathcal{D}^\mu,\eta^\nu_{\pm\boldsymbol{k}}\right]\right]-\left[\mathcal{D}_\mu,\left[\mathcal{D}^\nu,\eta^\mu_{\pm\boldsymbol{k}}\right]\right]+ig\left[\mathcal{F}_\mu{}^\nu,\eta^\mu_{\pm\boldsymbol{k}}\right]=0\\ \lim\limits_{t\to-\infty}\eta^\mu_{\pm\boldsymbol{k}}(t,\vec{\boldsymbol{x}})=\epsilon^\mu(\boldsymbol{k})\;e^{\pm ik\cdot x}\end{cases}$$

Single gluon spectrum at NLO

- For a small field fluctuation a^μ (not driven by a source) propagating on top of the classical field $\mathcal{A}^\mu$, one can prove :

$$a^\mu(x) = \Big[\int_{\vec{u}\in\mathrm{LC}} a(u)\cdot\mathbb{T}_{\boldsymbol{u}}\Big]\;\mathcal{A}^\mu(x)$$



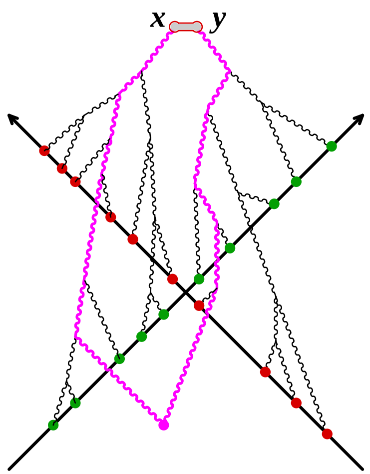
  - 'LC' denotes a surface just above the backward light-cone
  - $\mathbb{T}_{\boldsymbol{u}}$ is the generator of shifts of the initial value of the fields on this surface :

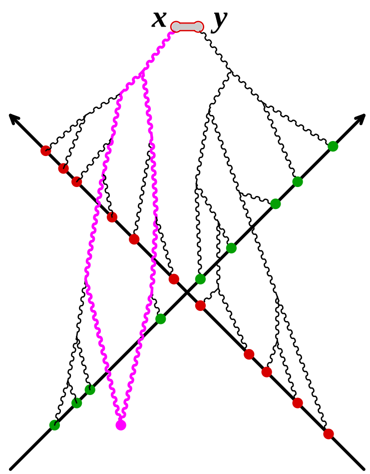
$$\mathcal{F}[\mathcal{A}+a] \equiv \exp\Big[\int_{\vec{u}\in\mathrm{LC}} a(u)\cdot\mathbb{T}_{\boldsymbol{u}}\Big]\;\mathcal{F}[\mathcal{A}]$$

Note : this construction is possible only because the objects involved in the problem obey retarded boundary conditions

Single gluon spectrum at NLO

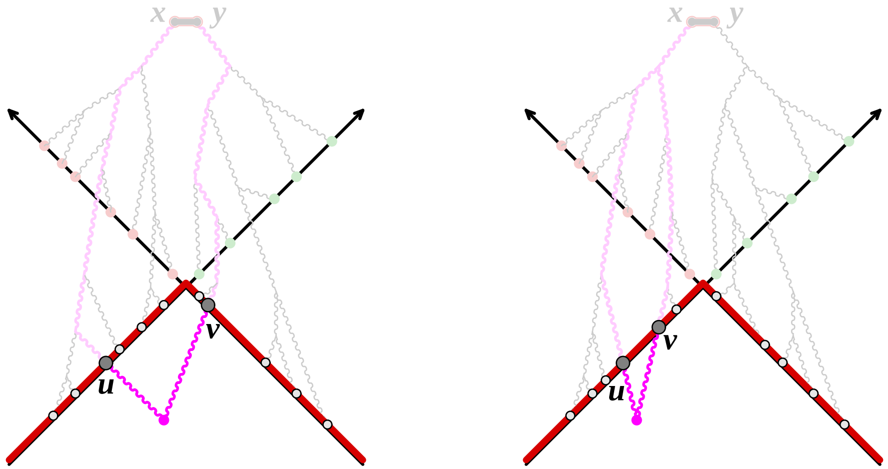
- 1-loop graphs contributing to the gluon spectrum at NLO :





Single gluon spectrum at NLO

- 1-loop graphs contributing to the gluon spectrum at NLO :



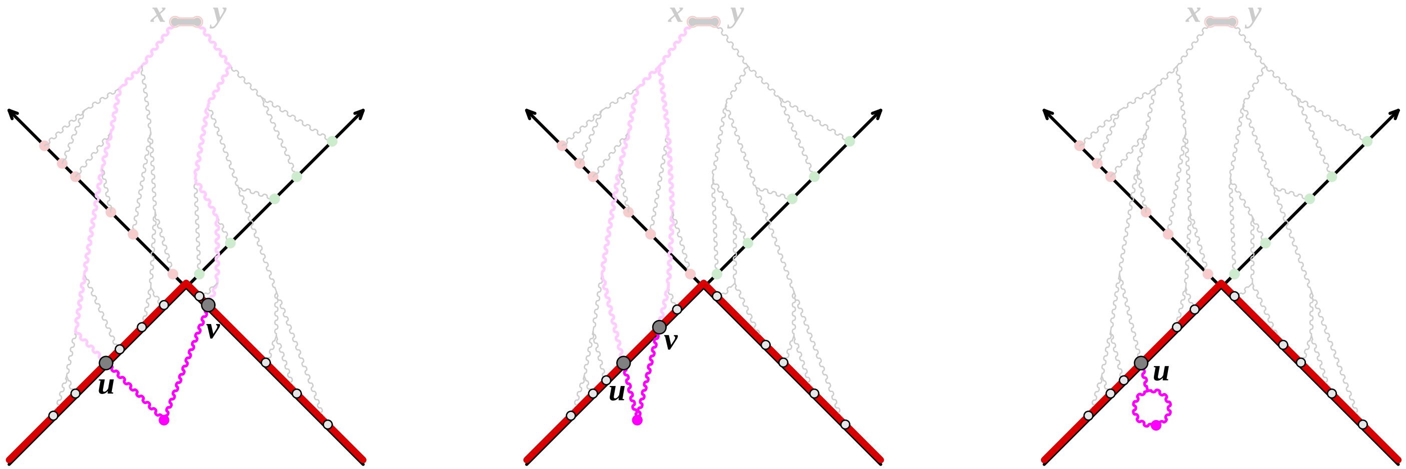
- They can be written as a perturbation of the LC initial fields :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in {\rm LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

$$\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \equiv \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\;\eta_{-\boldsymbol{k}}(u)\,\eta_{+\boldsymbol{k}}(v)$$

Single gluon spectrum at NLO

- 1-loop graphs contributing to the gluon spectrum at NLO :



- The loop correction can also be below the light-cone :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{NLO}} = \left[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\mathrm{LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}}$$

▷ the functions $\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ and $\beta(\vec{\boldsymbol{u}})$ can be evaluated analytically

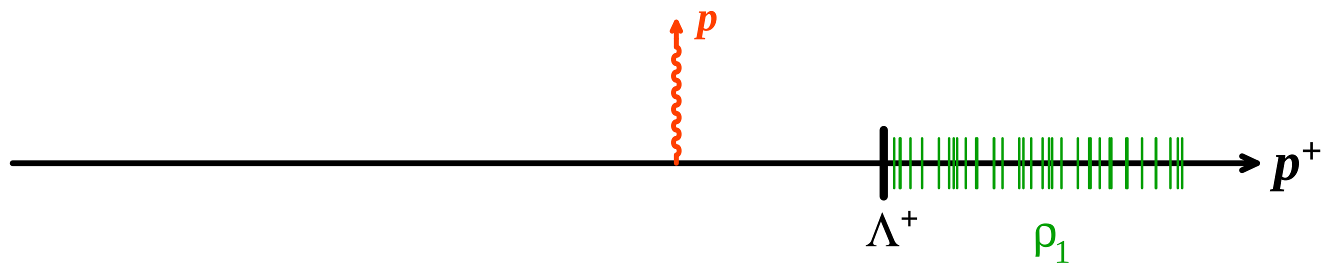
Divergences

- If $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ belong to the same branch of the LC (e.g. $u^- = v^- = \epsilon$), the function $\boldsymbol{\Sigma}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$ contains

$$\boldsymbol{\Sigma}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \sim \int_0^{+\infty} \frac{dk^+}{k^+} \cdots e^{ik^-(u^+-v^+)} \qquad \text{with} \quad k^- \equiv \frac{\boldsymbol{k}_\perp^2}{2k^+}$$

▷ the integral converges at $k^+ = 0$ but not when $k^+ \to +\infty$

Note : the log is a $\log(\Lambda^+/p^+)$, where Λ^+ is the boundary between the hard color sources and the fields, and p^+ the longitudinal momentum of the produced gluon



- Similar considerations apply when $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ both belong to the other branch of the LC, leading to a $\log(\Lambda^-/p^-)$

Leading Log approximation

- In the LC gauge $\mathcal{A}^+ = 0$, the operator $\eta(u) \cdot \mathbb{T}_{\boldsymbol{u}}$ is

$$\eta(u)\cdot\mathbb{T}_{\boldsymbol{u}} \equiv (\partial^- \eta_a^i(u)) \frac{\delta}{\delta(\partial^- \mathcal{A}_a^i(u))} + \eta_a^-(u) \frac{\delta}{\delta \mathcal{A}_a^-(u)} + (\partial_\mu \eta_a^\mu(u)) \frac{\delta}{\delta(\partial_\mu \mathcal{A}_a^\mu(u))}$$

- An explicit calculation of $\partial^- \eta^i_{\pm\boldsymbol{k}}$ and $\eta^-_{\pm\boldsymbol{k}}$ shows that these components have an extra $1/k^+$ when $k^+ \to +\infty$

- At leading log, it seems sufficient to consider :

$$\eta(u) \cdot \mathbb{T}_{\boldsymbol{u}} \underset{\text{LLog}}{=} (\partial_\mu \eta_a^\mu(u)) \frac{\delta}{\delta(\partial_\mu \mathcal{A}_a^\mu(u))}$$

This is almost correct, but not quite...

Leading Log approximation

- The space-time above the LC contains a classical background field,

$$\mathcal{A}^{\pm} = 0 \quad , \quad \mathcal{A}^{i} = \frac{i}{g}\Omega^{\dagger}\partial^{i}\Omega$$

▷ the interaction of the fluctuation with a background field can turn terms that are not divergent on the LC into divergent terms !
(factors of k^+ can arise in the 3-gluon derivative coupling)

- Because the background is a pure gauge, this problem is circumvented by using $\Omega_{ab}\eta_b$ instead of η_a as the initial condition :

$$\eta(u)\cdot\mathbb{T}_{\boldsymbol{u}} \equiv (\partial^{-}\Omega_{ab}\eta_b^{i}(u))\frac{\delta}{\delta(\partial^{-}\Omega_{ab}\mathcal{A}_b^{i}(u))} + \Omega_{ab}\eta_b^{-}(u)\frac{\delta}{\delta\Omega_{ab}\mathcal{A}_b^{-}(u)}$$
$$\underline{+(\partial_{\mu}\Omega_{ab}\eta_b^{\mu}(u))\frac{\delta}{\delta(\partial_{\mu}\Omega_{ab}\mathcal{A}_b^{\mu}(u))}}$$

▷ at leading log, only the last term matters

JIMWLK Hamiltonian

- The coefficient of the leading log does not depend on u^+, v^+
- Derivatives with respect to $\partial_\mu \Omega_{ab} \mathcal{A}_b^\mu(u)$ can be mapped to derivatives with respect to the slowest color sources :

$$\int du^+ \frac{\delta}{\delta(\partial_\mu \Omega_{ab} \mathcal{A}_b^\mu(u))} = \int d^2\vec{\boldsymbol{x}}_\perp \left\langle \vec{\boldsymbol{u}}_\perp \right| \frac{1}{\boldsymbol{\partial}_\perp^2} \left| \vec{\boldsymbol{x}}_\perp \right\rangle \frac{\delta}{\delta \widetilde{\mathcal{A}}_a^+(\epsilon, \vec{\boldsymbol{x}}_\perp)}$$

with $\quad -\boldsymbol{\partial}_\perp^2 \widetilde{\mathcal{A}}^+(\epsilon, \vec{\boldsymbol{x}}_\perp) = \rho(\epsilon, \vec{\boldsymbol{x}}_\perp)$

- When $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the same branch of the LC, we have

$$\frac{1}{2} \int_{\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathrm{LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}) \, \mathbb{T}_{\boldsymbol{u}} \mathbb{T}_{\boldsymbol{v}} \underset{\mathrm{LLog}}{=} \frac{1}{2} \log\left(\frac{\Lambda^+}{p^+}\right) \int_{\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp} \eta_{ab}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \frac{\delta^2}{\delta \widetilde{\mathcal{A}}_a^+(\epsilon, \vec{\boldsymbol{x}}_\perp) \delta \widetilde{\mathcal{A}}_b^+(\epsilon, \vec{\boldsymbol{y}}_\perp)}$$

with $\eta_{ab}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv$

$$\frac{1}{\pi} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp) \cdot (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}})^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2} \times \left[1 + \Omega(x)\Omega^\dagger(y) - \Omega(x)\Omega^\dagger(z) - \Omega(z)\Omega^\dagger(y)\right]_{ab}$$

JIMWLK Hamiltonian

- In principle, one could evaluate the term involving $\beta(\vec{\boldsymbol{u}})$ by solving explicitly its EOM
- Shortcut: by using the Green's formula for this fluctuation, one can show directly that

$$\int_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}} \underset{\mathrm{LLog}}{=} \frac{1}{2}\log\left(\frac{\Lambda^+}{p^+}\right)\int_{\vec{\boldsymbol{x}}_\perp}\left(\int_{\vec{\boldsymbol{y}}_\perp}\frac{\delta\eta_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{\boldsymbol{y}}_\perp)}\right)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{\boldsymbol{x}}_\perp)}$$

- Combining the real and virtual terms :

$$\left[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in\mathrm{LC}} \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right]$$

$$\underset{\mathrm{LLog}}{=} \log\left(\frac{\Lambda^+}{p^+}\right)\underbrace{\frac{1}{2}\int_{\vec{\boldsymbol{y}}_\perp}\frac{\delta}{\delta\widetilde{\mathcal{A}}_b^+(\epsilon,\vec{\boldsymbol{y}}_\perp)}\eta_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta}{\delta\widetilde{\mathcal{A}}_a^+(\epsilon,\vec{\boldsymbol{x}}_\perp)}}_{\text{JIMWLK }\mathcal{H}}$$

Leading Log divergences

- The configuration where $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the first branch of the LC can be rewritten as

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} \underset{\rm LLog}{=} \log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 \; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

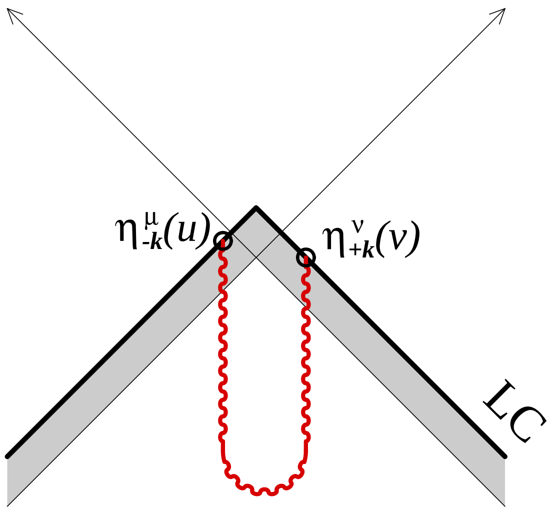
with $\mathcal{H}_1$ the JIMWLK Hamiltonian for the first nucleus

- Including also the configuration where both $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the second branch of the LC, we get

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} \underset{\rm LLog}{=} \left[\log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 + \log\left(\frac{\Lambda^-}{p^-}\right) \mathcal{H}_2\right] \; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

Leading Log divergences

- The only remaining possibility is to have $\vec{u}$ and $\vec{v}$ on different branches of the LC



However, there is no log divergence in this case, since the k^+ integral is of the form :

$$\int \frac{dk^+}{k^+} \cdots e^{ik^+(u^- - v^-)}\; e^{ik^-(u^+ - v^+)}$$

$\triangleright$ no mixing of the divergences of the two nuclei

Leading Log factorization

- All the above discussion is for one given configuration of the sources $\rho_{1,2}$ (or of the fields $\widetilde{\mathcal{A}}^{\pm}_{1,2}$). Averaging over all the configurations of the sources in the two projectiles, and using the hermiticity of the JIMWLK Hamiltonian, we get

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle_{\mathrm{LO+NLO}} \underset{\mathrm{LLog}}{=} \int \left[D\widetilde{\mathcal{A}}_1^+ \, D\widetilde{\mathcal{A}}_2^- \right] \times \left(\left[1 + \log\left(\frac{\Lambda^+}{p^+}\right) \mathcal{H}_1 + \log\left(\frac{\Lambda^-}{p^-}\right) \mathcal{H}_2 \right] W[\widetilde{\mathcal{A}}_1^+] \, W[\widetilde{\mathcal{A}}_2^-] \right) \left. \frac{dN}{d^3\vec{\boldsymbol{p}}} \right|_{\mathrm{LO}}$$

- This is a 1-loop result. Using RG arguments, this leads to the following factorized formula for the resummation of the leading log terms to all orders :

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle \underset{\mathrm{LLog}}{=} \int \left[D\widetilde{\mathcal{A}}_1^+ \, D\widetilde{\mathcal{A}}_2^- \right] W_{Y_1}[\widetilde{\mathcal{A}}_1^+] \, W_{Y_2}[\widetilde{\mathcal{A}}_2^-] \left. \frac{dN}{d^3\vec{\boldsymbol{p}}} \right|_{\mathrm{LO}}$$

with $\displaystyle \frac{\partial}{\partial Y} W_Y = \mathcal{H}\, W \quad , \quad Y_1 = \log(P_1^+/p^+) \quad , \quad Y_2 = \log(P_2^-/p^-)$

Initial conditions for hydro

- Instead of the single gluon spectrum which is not of direct interest in practical applications, one can reproduce all the above discussion for the energy-momentum tensor :

$$\left\langle T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)\right\rangle \underset{\mathrm{LLog}}{=} \int \left[D\widetilde{\mathcal{A}}_1^+\, D\widetilde{\mathcal{A}}_2^-\right] W_{Y_1}\left[\widetilde{\mathcal{A}}_1^+\right] W_{Y_2}\left[\widetilde{\mathcal{A}}_2^-\right] \Big[T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)\Big]_{\mathrm{LO}}$$

with $\quad Y_1 = Y_{\mathrm{beam}} - \eta \quad , \quad Y_2 = Y_{\mathrm{beam}} + \eta$

- How to do it in practice :
  - Construct initial conditions for $W[\widetilde{\mathcal{A}}^\pm]$, e.g. from the MV model
  - Solve the JIMWLK equation (see Rummukainen, Weigert (2003)) in order to get ensembles $\{\widetilde{\mathcal{A}}^+\}_{Y_1}, \{\widetilde{\mathcal{A}}^-\}_{Y_2}$ of field configurations at each rapidity
  - At each rapidity η, chose randomly an element $\widetilde{\mathcal{A}}_1^+$ from the ensemble $\{\cdot\}_{Y_{\mathrm{beam}}-\eta}$ and an element $\widetilde{\mathcal{A}}_2^-$ from $\{\cdot\}_{Y_{\mathrm{beam}}+\eta}$
  - Compute $T^{\mu\nu}(\tau,\eta,\vec{\boldsymbol{x}}_\perp)$ for this pair $(\widetilde{\mathcal{A}}_1^+, \widetilde{\mathcal{A}}_2^-)$
  - Repeat until the Monte-Carlo integral converges

Initial conditions for hydro

- The previous procedure will only reproduce the energy momentum tensor profile for the average event. Ideally, one would like to be able to generate “events”, i.e. a collection of energy-momentum profiles, each of them representing the outcome of one collision

- In order to generate events with the correct initial 2-point correlations, one would need to compute

$$\big\langle T^{\mu\nu}(x)T^{\rho\sigma}(y)\big\rangle - \big\langle T^{\mu\nu}(x)\big\rangle\big\langle T^{\rho\sigma}(y)\big\rangle$$

- Origin of the fluctuations :
  - Distribution of the nucleons inside the nucleus
  - Distribution of the color charges inside each nucleon
  - The previous two being fixed, there is a fluctuation of the distribution of produced particles in the collision. Power counting arguments show that this correlation is as large as the mean

Summary

Summary

- The single gluon spectrum at LO involves only retarded classical fields
- Factorization works (as expected) for the single inclusive gluon spectrum
- With a complete knowledge of the 1- and 2-gluon initial spectra, one could in principle build a CGC-based event generator for AA collisions, that has the correct correlations up to 2 particles (but not beyond that)
- Interesting issues :
  - 2-gluon spectrum and factorization thereof
  - Role of the instability of the classical field in thermalization?
  - Factorization at Next-to-Leading Log?
  - What about exclusive quantities, like diffraction and rapidity gaps?

Extra bits

Generating functional

- Consider a function $z(\vec{\boldsymbol{p}})$, and define the functional

$$F[z] \equiv \frac{1}{n!} \sum_{n=0}^{+\infty} \int d\Phi_1 \cdot d\Phi_n \; z(\vec{\boldsymbol{p}}_1) \cdots z(\vec{\boldsymbol{p}}_n) \; \left| \left\langle \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_{n\,\mathrm{out}} \middle| 0_{\mathrm{in}} \right\rangle \right|^2$$

- Any physical quantity can be obtained from $F[z]$
  - Single inclusive spectrum :

$$\frac{d\overline{N}}{d^3\vec{\boldsymbol{p}}} = \left. \frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} \right|_{z=1}$$

  - Double inclusive spectrum (correlated part) :

$$\mathcal{C}(\vec{\boldsymbol{p}}_1, \vec{\boldsymbol{p}}_2) = \left. \frac{\delta^2 F[z]}{\delta z(\vec{\boldsymbol{p}}_1)\delta z(\vec{\boldsymbol{p}}_2)} - \frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}}_1)} \frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}}_2)} \right|_{z=1}$$

From F[z] to an event generator

- Discretize the phase-space by dividing it in bins 1 to N. The functional $F[z(\vec{\boldsymbol{p}})]$ becomes a function of N variables $F(\boldsymbol{z})$ with $\boldsymbol{z} \equiv (z_1 \cdots z_N)$
- It can be parameterized by $\ln F(\boldsymbol{z}) = \sum_{\boldsymbol{r}} b(\boldsymbol{r})(\boldsymbol{z}^{\boldsymbol{r}} - 1)$, where the sum is over vectors $\boldsymbol{r}$ with integer components ≥ 1
- If $\boldsymbol{n} \equiv (n_1 \cdots n_N)$, the probability $P(\boldsymbol{n})$ to have n_1 particles in bin 1, etc... can be calculated in terms of the $b(\boldsymbol{r})$ (complicated but explicit formulas) $\triangleright$ generate events according to this probability law
- Note : moments of the distribution are related to the $b(\boldsymbol{r})$ by

$$\langle \boldsymbol{n}_i \rangle = \sum_{\boldsymbol{r}} \boldsymbol{r}_i \, b(\boldsymbol{r})$$

$$\langle \boldsymbol{n}_i \boldsymbol{n}_j \rangle - \langle \boldsymbol{n}_i \rangle \langle \boldsymbol{n}_j \rangle = \sum_{\boldsymbol{r}} \boldsymbol{r}_i \boldsymbol{r}_j \, b(\boldsymbol{r})$$

$\triangleright$ one could truncate the series of the $b(\boldsymbol{r})$ to have a generator that reproduces all the correlations up to a certain order

From F[z] to an event generator

- Example : the simplest truncation is to assume that the n_i's are independent Poissonian variables :

$$\langle \boldsymbol{n}_i \boldsymbol{n}_j \rangle - \langle \boldsymbol{n}_i \rangle \langle \boldsymbol{n}_j \rangle = 0 \quad \text{for} \quad i \neq j$$
$$\langle \boldsymbol{n}_i^2 \rangle - \langle \boldsymbol{n}_i \rangle^2 = \langle \boldsymbol{n}_i \rangle^2$$

- This is realized if

$$b(\boldsymbol{r}) = 0 \quad \text{if at least one component of } \boldsymbol{r} \text{ is } \geq 1$$
$$b(\boldsymbol{r}) = 0 \quad \text{if at least two components of } \boldsymbol{r} \text{ are } \neq 0$$

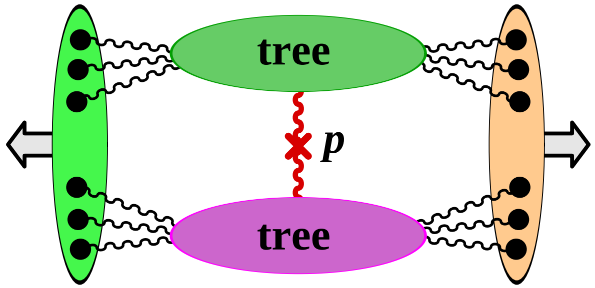
▷ in this approximation, the only nonzero $b(\boldsymbol{r})$'s are the $b(0 \cdots 1 \cdots 0)$. They can be determined from the inclusive 1-gluon spectrum

- If in addition, one knew the inclusive 2-gluon spectrum, one could make a less drastic truncation, that would provide a generator that reproduces all the 2-particle correlations

F[z] at Leading Order

- Could one evaluate directly $F[z(\vec{\boldsymbol{p}})]$?
- At leading order, this is given by the same topologies of diagrams as those involved in $\overline{N}$:

$$\frac{1}{F[z]}\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



but the internal cut propagators are multiplied by $z(\vec{\boldsymbol{p}})$

- One can also write it in terms of two fields $\mathcal{A}_\pm(x)$ as :

$$\frac{1}{F[z]}\frac{\delta F[z]}{\delta z(\vec{\boldsymbol{p}})} = \int_{x,y} e^{ip\cdot(x-y)} \cdots \mathcal{A}_+^\mu(x)\mathcal{A}_-^\nu(y)$$

Note : this formula is formally identical to the formula for the inclusive spectrum, but the fields $\mathcal{A}_\pm$ it contains are different

1-gluon spectrum boundary conditions

- For later comparison, it is useful to rewrite the boundary conditions for the single inclusive gluon spectrum in terms of the Fourier modes of the fields :

$$\mathcal{A}_\epsilon(x) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[a_\epsilon^{(+)}(x_0,\vec{\boldsymbol{p}})\, e^{-ip\cdot x} + a_\epsilon^{(-)}(x_0,\vec{\boldsymbol{p}})\, e^{+ip\cdot x} \right]$$

- The retarded boundary conditions are equivalent to :

$$a_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$
$$a_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = a_+^{(+)}(+\infty,\vec{\boldsymbol{p}})$$
$$a_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

F[z] at Leading Order

- $\mathcal{A}_\pm$ are still solutions of the classical EOMs
- The boundary conditions can be rewritten in terms of the Fourier coefficients as :

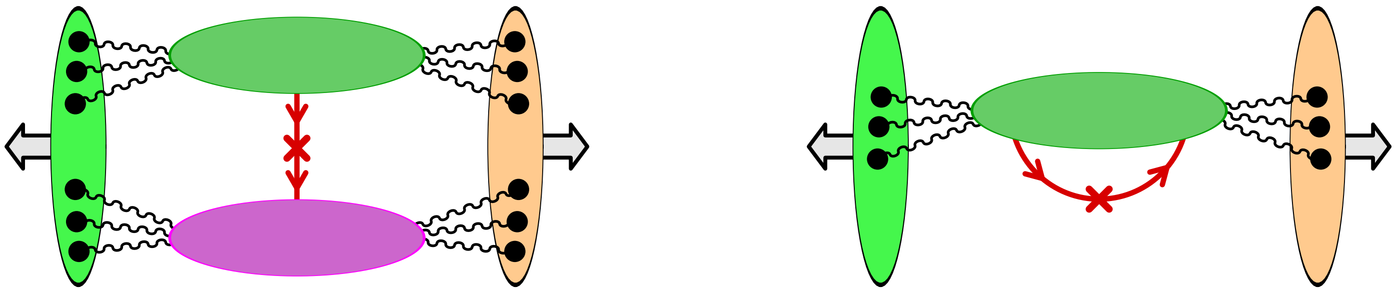
$$
\begin{aligned}
a_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) &= a_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0 \\
a_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) &= z(\vec{\boldsymbol{p}})\; a_+^{(+)}(+\infty,\vec{\boldsymbol{p}}) \\
a_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) &= z(\vec{\boldsymbol{p}})\; a_-^{(-)}(+\infty,\vec{\boldsymbol{p}})
\end{aligned}
$$

- The function $z(\vec{\boldsymbol{p}})$ enters only via the boundary conditions
- The only difference at leading order between inclusive quantities and more exclusive ones comes from the boundary conditions
- These boundary conditions are not retarded
  - ▷ extremely difficult to solve numerically

Inclusive quark spectrum

FG, Kajantie, Lappi (2004, 2005)

- One can construct for quarks an operator $\mathcal{C}_\mathrm{q}$ that plays the same role as $\mathcal{C}$ for the gluons
- By repeating the same arguments, we find two generic topologies contributing to the inclusive quark spectrum :



(the blobs are sums of cut diagrams)

- The first topology cannot exist because the quark line is not closed on itself

  ▷ the quark spectrum starts at one loop

Quark production at one loop

- At lowest order (one loop), the quark spectrum reads :

$$\frac{d\overline{N}_{\rm q}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot x}\,\overline{u}(\vec{\boldsymbol{p}})\,(i\overrightarrow{\not\partial}_x - m)\,S_{+-}(x,y)\,(i\overleftarrow{\not\partial}_y + m)\,u(\vec{\boldsymbol{p}})\,e^{-ip\cdot y}$$

where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

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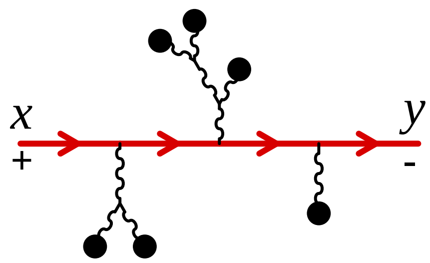
where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

- We need to calculate the sum of the following tree diagrams :



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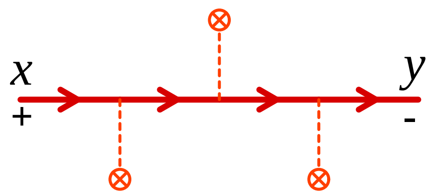
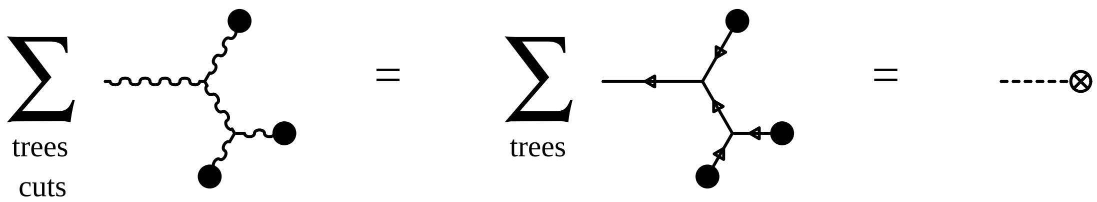


- Perform a resummation of all the sub-diagrams that correspond to the retarded classical solution :



Quark propagator

- The summation of all the classical field insertions can be done by solving a Lippmann-Schwinger equation :

$$S_{\epsilon\epsilon'}(x,y) = S^0_{\epsilon\epsilon'}(x,y) - ig \sum_{\eta=\pm} (-1)^\eta \int d^4z\, S^0_{\epsilon\eta}(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_{\eta\epsilon'}(z,y)$$

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the $\pm$ indices :

$$S_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{S}_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon} U_{\beta\epsilon'} S_{\epsilon\epsilon'}$$

$$(-1)^\epsilon \delta_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \sum_{\epsilon=\pm} U_{\alpha\epsilon} U_{\beta\epsilon} (-1)^\epsilon$$

- A useful choice for the rotation matrix U is $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Quark propagator

- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{S}_{\alpha\beta} = \begin{pmatrix} 0 & S_A \\ S_R & S_D \end{pmatrix} \quad , \qquad \boldsymbol{\Sigma}_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $S_D^0(p) = 2\pi(\not{p} + m)\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{S}^0\boldsymbol{\Sigma}$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that S_R and S_A do not mix, i.e. they obey equations such as :

$$S_R(x,y) = S_R^0(x,y) - i\,g\int d^4z\, S_R^0(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_R(z,y)$$

- One can solve S_D in terms of S_R and S_A :

$$S_D = S_R * S_R^{0\,-1} * S_D^0 * S_A^{0\,-1} * S_A$$

Quark propagator

- In order to go back to S_{+-}, invert the rotation :

$$S_{+-} = \frac{1}{2}\left[S_{A} - S_{R} - S_{D}\right]$$

- At this point, we can rewrite the quark spectrum in terms of retarded and advanced quark propagators in the classical background

- Finally, one can rewrite it in terms of retarded solutions of the Dirac equation on top of the background $\mathcal{A}_\mu(x)$

$$\frac{d\overline{N}_{\rm q}}{dY d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int\frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\Big|\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}})\Big|^2$$

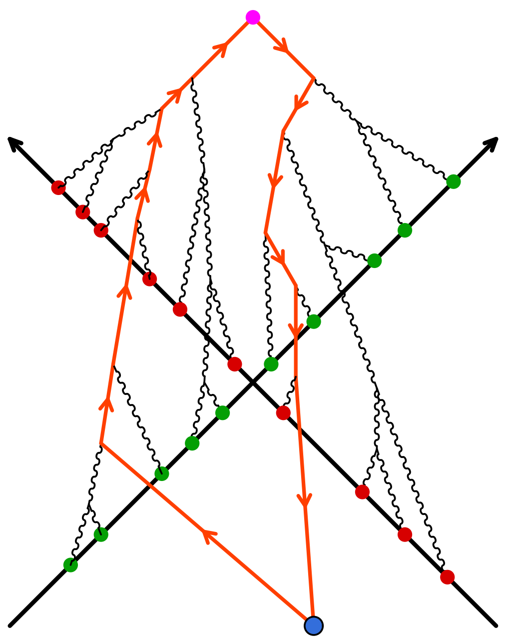
with

$$\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}}) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\; u^\dagger(\vec{\boldsymbol{p}})\psi_{\boldsymbol{q}}(x)$$

$$(i\partial\!\!\!/_x - g\mathcal{A}\!\!\!/(x) - m)\psi_{\boldsymbol{q}}(x) = 0\;,\quad \psi_{\boldsymbol{q}}(x^0,\vec{\boldsymbol{x}}) \underset{x^0\to-\infty}{=} v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

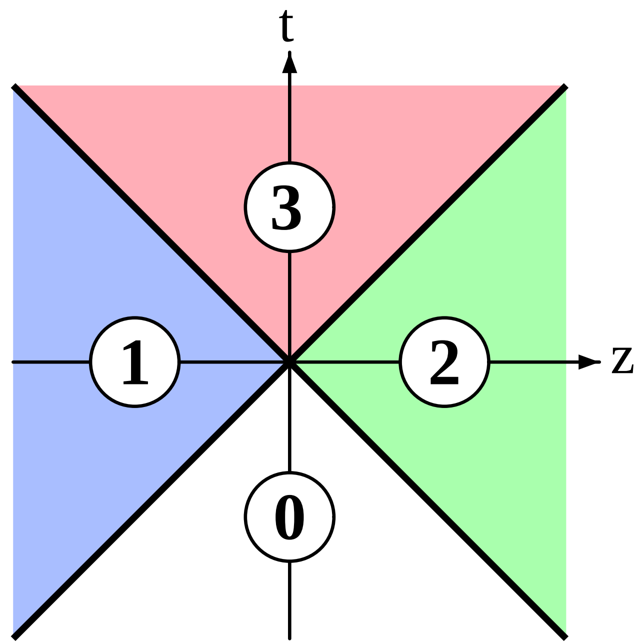
Quark propagator

- This calculation amounts to summing the following diagrams :



Background field

- Space-time structure of the classical color field:

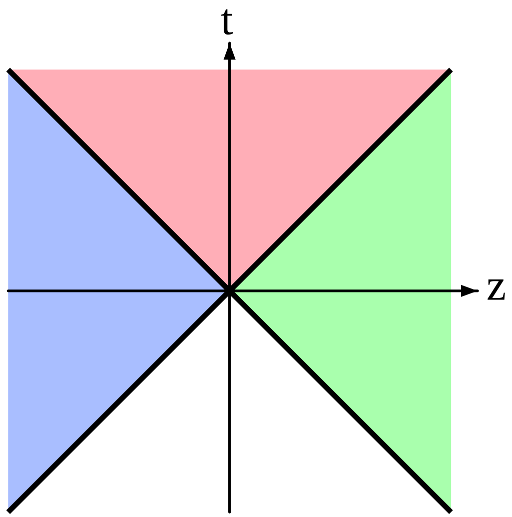


- Region 0: $\mathcal{A}^\mu = 0$
- Region 1: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 2: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 3: $\mathcal{A}^\mu \neq 0$

- **Notes:**
  - In the region 3, $\mathcal{A}^\mu$ is known only numerically
  - We must solve the Dirac equation numerically as well

Quark propagation

- Propagation through region 0:

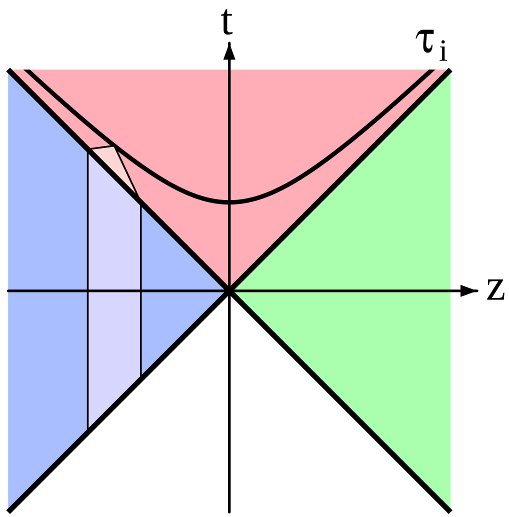


▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Quark propagation

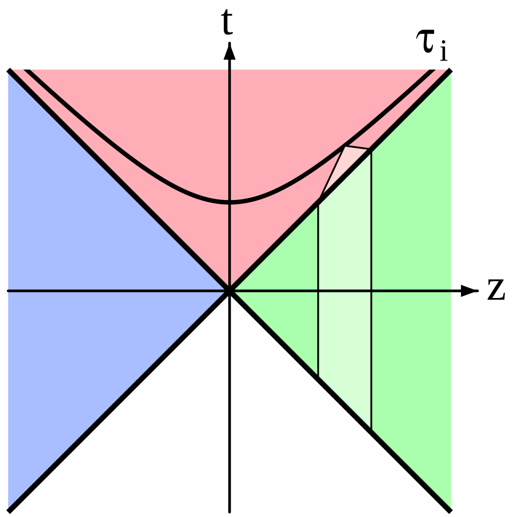
- Propagation through region 1:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},1}(\tau_i)$ can be obtained analytically

Quark propagation

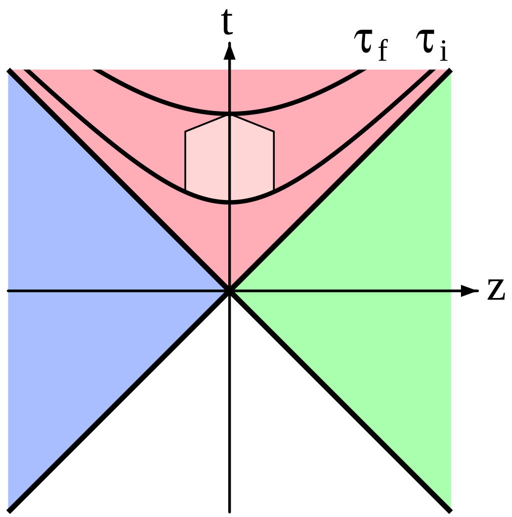
- Propagation through region 2:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},2}(\tau_i)$ can be obtained analytically

Quark propagation

- Propagation through region 3:



▷ One must solve the Dirac equation :

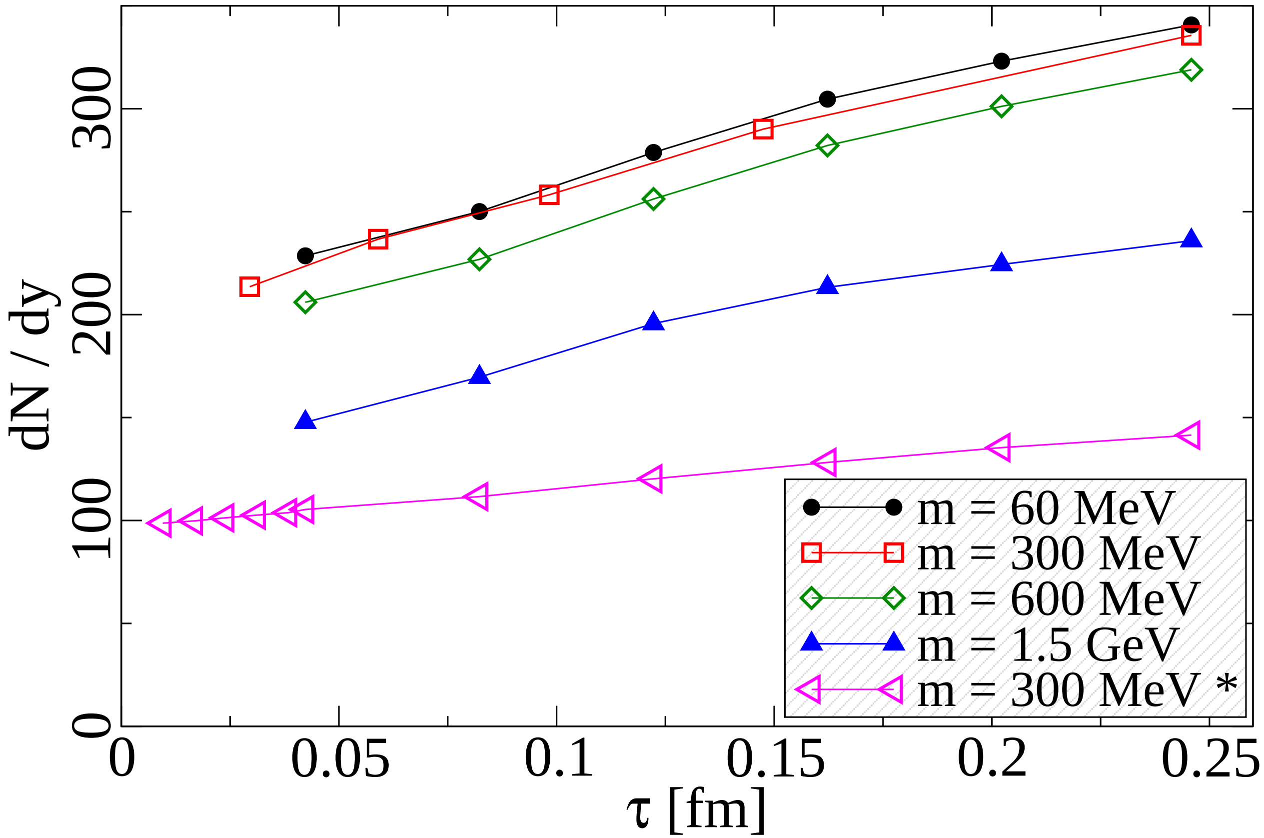
$$\left[i\partial\!\!\!/ - g\mathcal{A}\!\!\!/ - m\right] \psi_{\boldsymbol{q}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi_{\boldsymbol{q},1}(\tau_i) + \psi_{\boldsymbol{q},2}(\tau_i)$

($\tau_i = 0^+$ in practice)

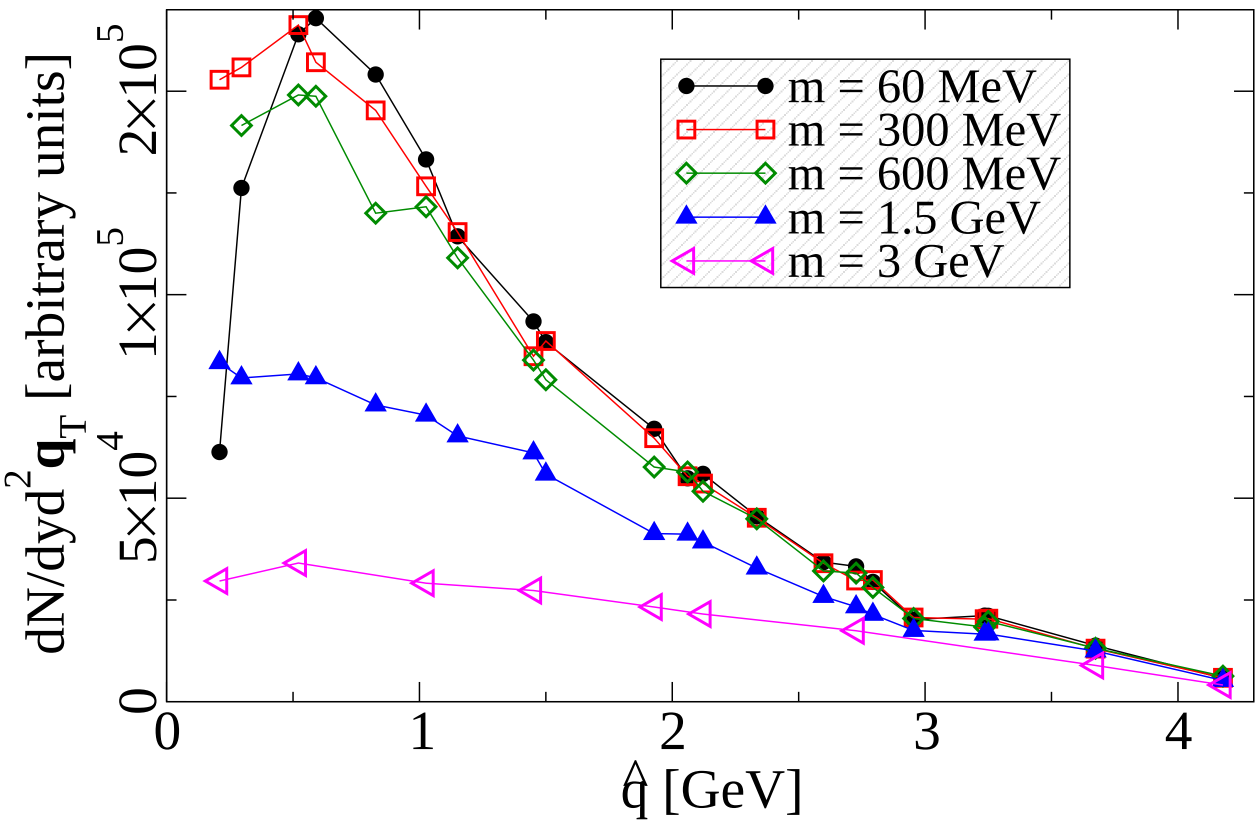
Time dependence

- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



Spectra for various quark masses

- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :



Longitudinal expansion

- For a system finite in the η direction, the gluons will have a longitudinal velocity tied to their space-time rapidity

Longitudinal expansion

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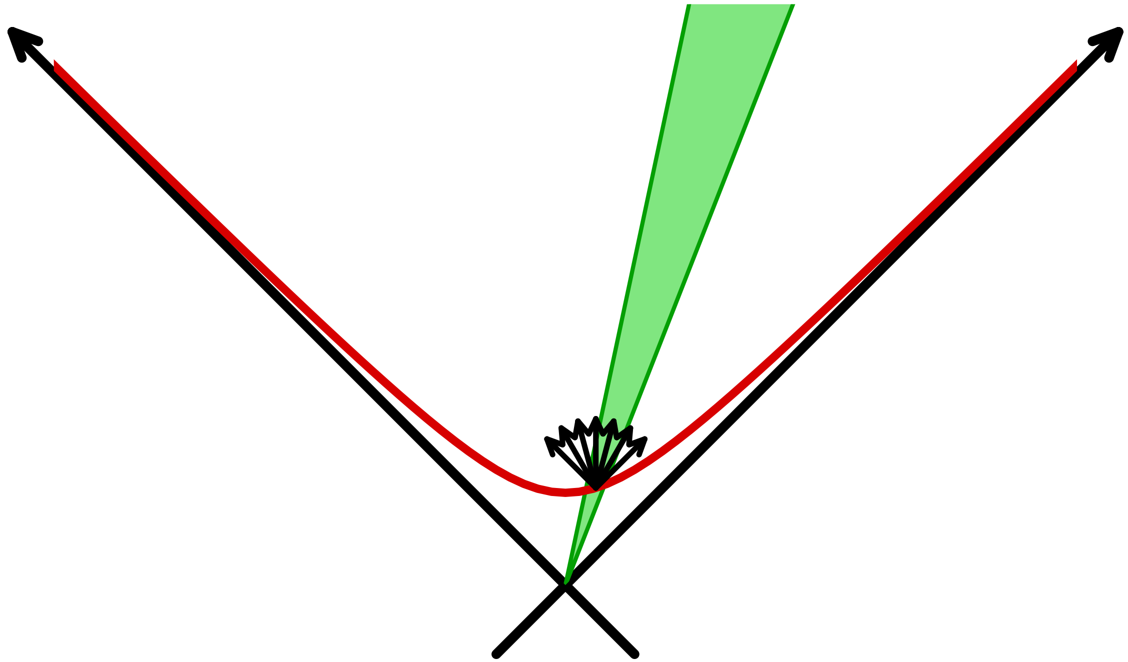

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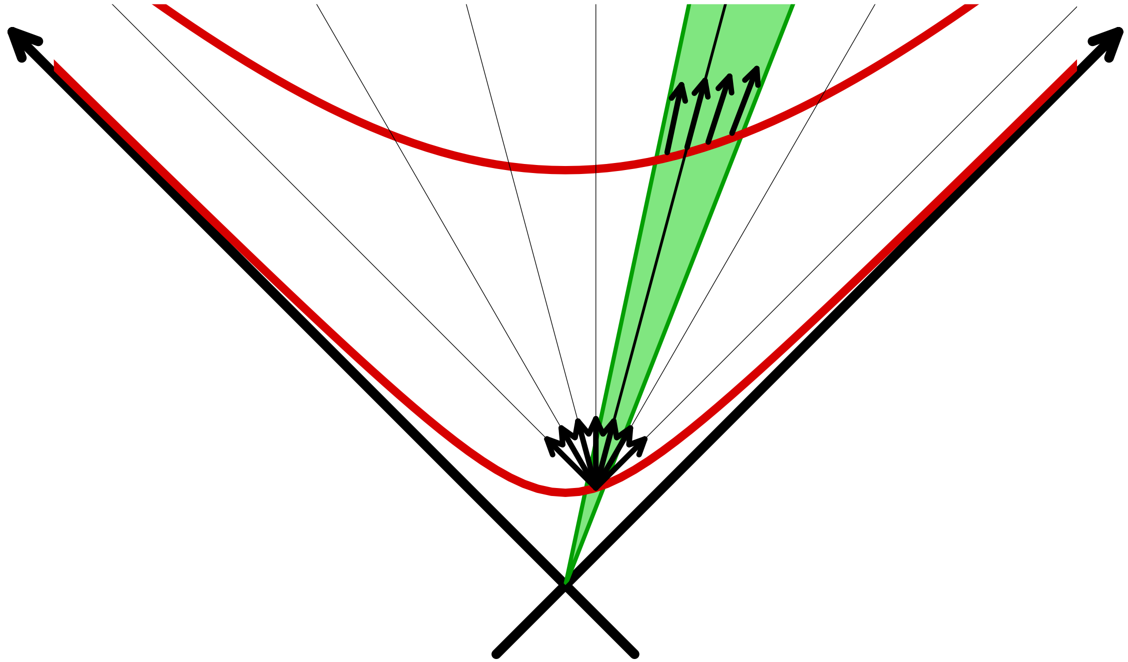


▷ at late times : if particles fly freely, only one longitudinal velocity can exist at a given η : $v_z = \tanh(\eta)$

▷ the expansion of the system is the main obstacle to local isotropy

Generating function

- Let P_n be the probability of producing n particles
- Define the generating function :

$$F(z) \equiv \sum_{n=0}^{\infty} P_n \, z^n$$

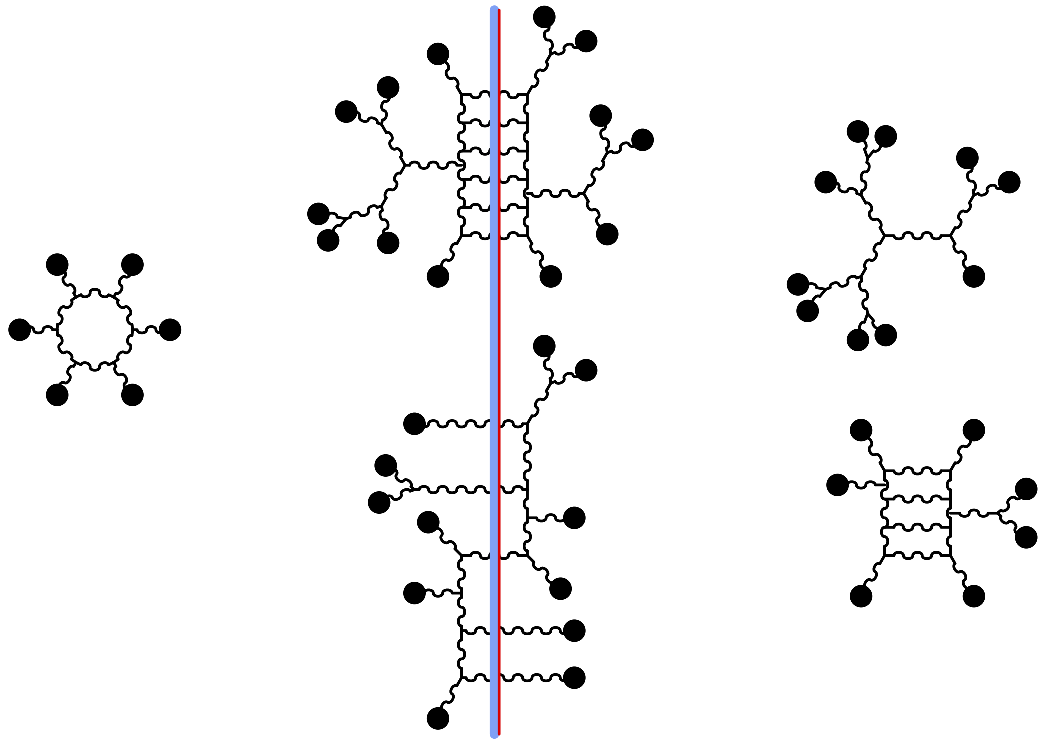
- From unitarity, $F(1) = \sum_{n=0}^{\infty} P_n = 1$. Thus, we can write

$$\ln(F(z)) \equiv \sum_{r=1}^{\infty} b_r \, (z^r - 1)$$

- At the moment, we need to know only very little about the b_r :
  - $F(z)$ is a sum of diagrams that may or may not be connected
  - $\ln(F(z))$ involves only connected diagrams. Hence, the b_r's are given by certain sums of connected diagrams
  - Every diagram in b_r produces r particles

Generating function

- Example : typical term in the coefficient of z^{11}, with contributions from b_5 and b_6 :



Distribution of connected subdiagrams

- From this form of the generating function, one gets :

$$P_n = \sum_{p=0}^{n} \underbrace{e^{-\sum_r b_r} \frac{1}{p!} \sum_{\alpha_1+\cdots+\alpha_p=n} b_{\alpha_1} \cdots b_{\alpha_n}}_{\text{probability of producing } n \text{ particles in } p \text{ cut subdiagrams}}$$

- Summing on n, we get the probability of p cut subdiagrams :

$$R_p = \frac{1}{p!} \left[\sum_{r=1}^{\infty} b_r \right]^p e^{-\sum_r b_r}$$

Note : Poisson distribution of average $\langle N_{\text{subdiagrams}} \rangle = \sum_r b_r$

- By expanding the exponential, we get the probability of having p cut subdiagrams out of a total of m :

$$R_{p,m} = \frac{(-1)^{m-p}}{(m-p)!\,p!} \left[\sum_{r=1}^{\infty} b_r \right]^m$$

AGK identities

- The quantities $R_{p,m}$ obey the following relations :

$$\forall m \geq 2 \; , \quad \sum_{p=1}^{m} p \, R_{p,m} = 0 \; ,$$

$$\forall m \geq 3 \; , \quad \sum_{p=1}^{m} p(p-1) \, R_{p,m} = 0 \; , \cdots$$

- Interpretation : contributions with more than 1 subdiagram cancel in the average number of cut subdiagrams, etc...
- Correspondence with the original relations by Abramovsky-Gribov-Kancheli :
  - The original derivation is formulated in the framework of reggeon effective theories
  - Dictionary: *reggeon* $\longrightarrow$ *subdiagram*
  - These identities are more general than “reggeons”, and are valid for any kind of subdiagrams

Limitations

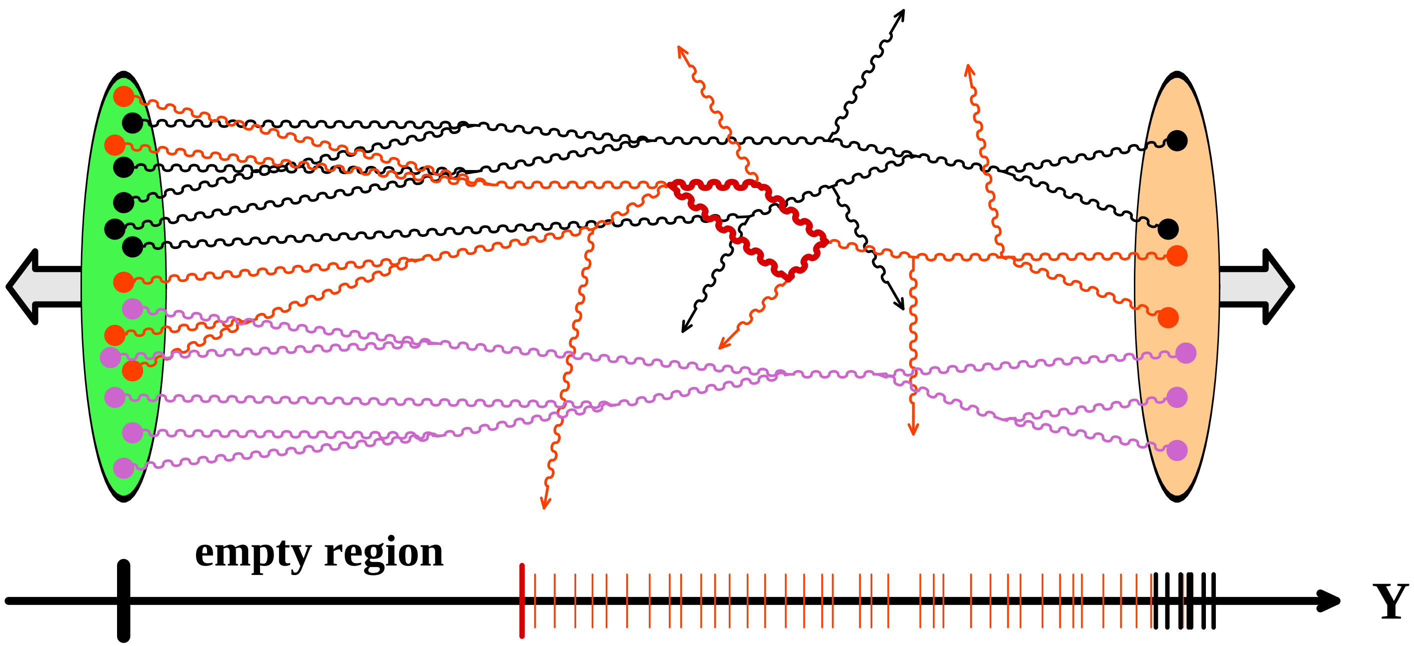
- The AGK relations, obtained by "integrating out" the number of produced particles, describe the combinatorics of connected diagrams
  - ▷ by doing that, a lot of information has been discarded
- For instance, to compute the average number of produced particles, one would write :

$$\langle n\rangle = \underbrace{\Big\langle N_{\rm subdiagrams}\Big\rangle}_{\sum_r b_r} \times \underbrace{\Big\langle \#\ \text{of particles per diagram}\Big\rangle}_{\text{requires a more detailed description}}$$

Exclusive processes

- So far, we have considered only inclusive quantities – i.e. the P_n are defined as probabilities of producing particles anywhere in phase-space

- What about events where a part of the phase-space remains unoccupied ? e.g. rapidity gaps



Main issues

1. How do we calculate the probabilities $P_n^{\rm excl}$ with an excluded region in the phase-space ?
   Can one calculate the total gap probability $P_{\rm gap} = \sum_n P_n^{\rm excl}$?
2. What is the appropriate distribution of sources $W_Y^{\rm excl}[\rho]$ to describe a projectile that has not broken up ?
3. How does it evolve with rapidity ?

   See : Hentschinski, Weigert, Schafer (2005)
4. Are there some factorization results, and for which quantities do they hold ?

Exclusive probabilities

- The probabilities $P_n^{\rm excl}[\boldsymbol{\Omega}]$, for producing n particles – only in the region $\boldsymbol{\Omega}$ – can also be constructed from the vacuum diagrams, as follows :

$$P_n^{\rm excl}[\boldsymbol{\Omega}] = \frac{1}{n!}\, \mathcal{C}_{\boldsymbol{\Omega}}^n \; e^{iV}\; e^{-iV^*}$$

where $\mathcal{C}_{\boldsymbol{\Omega}}$ is an operator that removes two sources and links the corresponding points by a cut (on-shell) line, for which the integration is performed only in the region $\boldsymbol{\Omega}$

- One can define a generating function,

$$F_{\boldsymbol{\Omega}}(z) \equiv \sum_n P_n^{\rm excl}[\boldsymbol{\Omega}]\; z^n \quad ,$$

whose derivative is given by the same diagram topologies as the derivative of the generating function for inclusive probabilities

Exclusive probabilities

- The generating function $F_\Omega(z)$ is a particular case of the functional $F[z(\vec{\boldsymbol{p}})]$ introduced earlier :

$$F_\Omega(z) = F[z(\vec{\boldsymbol{p}})] \quad \text{with} \quad \begin{cases} z(\vec{\boldsymbol{p}}) = z & \text{if } \vec{\boldsymbol{p}} \in \Omega \\ z(\vec{\boldsymbol{p}}) = 0 & \text{if } \vec{\boldsymbol{p}} \notin \Omega \end{cases}$$

- At leading order, the dependence on the region $\boldsymbol{\Omega}$ only affects the boundary conditions for the classical fields in terms of which one can write $F'_\Omega(z)/F_\Omega(z)$

- The integration constant needed to go from $F'_\Omega(z)/F_\Omega(z)$ to $F_\Omega(z)$ is non-trivial :

   ▷ $F_\Omega(1)$ – the sum of all the exclusive probabilities – is the probability of not having particles in the complement of $\boldsymbol{\Omega}$, i.e. the gap probability

Survival probability

- We can write :

$$F_\Omega(z) = F_\Omega(1) \; \exp\left\{ \int_1^z d\tau \, \frac{F'_\Omega(\tau)}{F_\Omega(\tau)} \right\}$$

▷ the prefactor $F_\Omega(1)$ will appear in all the exclusive probabilities

- This prefactor is nothing but the famous "survival probability" for a rapidity gap

▷ One can in principle calculate it by the general techniques developed for calculating inclusive probabilities :

$$F_\Omega(1) \; = \; F^{\rm incl}_{1-\Omega}(0)$$

▷ Note : it is incorrect to say that a certain process with a gap can be calculated by multiplying the probability of this process without the gap by the survival probability

Factorization ?

- In order to discuss factorization for exclusive quantities, one must calculate their 1-loop corrections, and study the structure of the divergences... Not done yet.
- Except for the case of Deep Inelastic Scattering, nothing is known regarding factorization for exclusive processes in a high density environment
- For the overall framework to be consistent, one should have factorization between the gap probability, $F_\Omega(1)$, and the source density studied in Hentschinski, Weigert, Schafer (2005) (and the ordinary $W_Y[\rho]$ on the other side)
- The total gap probability is the "most inclusive" among the exclusive quantities one may think of. For what quantities – if any – does factorization work ?

Action of Tu

- One can prove the following formula for the evolution of a small perturbation $a(x)$ on top of the classical field :

$$a(x) \quad = \int_{\vec{\boldsymbol{u}}\in \mathrm{LC}} \Big[a_{\rm in}(\vec{\boldsymbol{u}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\,\mathcal{A}(x)$$

$\triangleright$ from the classical field $\mathcal{A}(x)$, the operator $a_{\rm in}(\vec{\boldsymbol{u}})\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}$ builds the fluctuation $a(x)$ whose initial condition on the light-cone is $a_{\rm in}(\vec{\boldsymbol{u}})$

- From this formula, one sees that $\boldsymbol{T}_{\vec{\boldsymbol{u}}}\mathcal{A}(x)$ determines how the fluctuation $a(x)$ is sensitive on its initial condition

- If there is an instability in the system, two fluctuations whose value on the light-cone differ by an amount $\sim \epsilon$ will differ by an amount $\sim \epsilon \exp(\#\sqrt{\tau})$ at the time τ

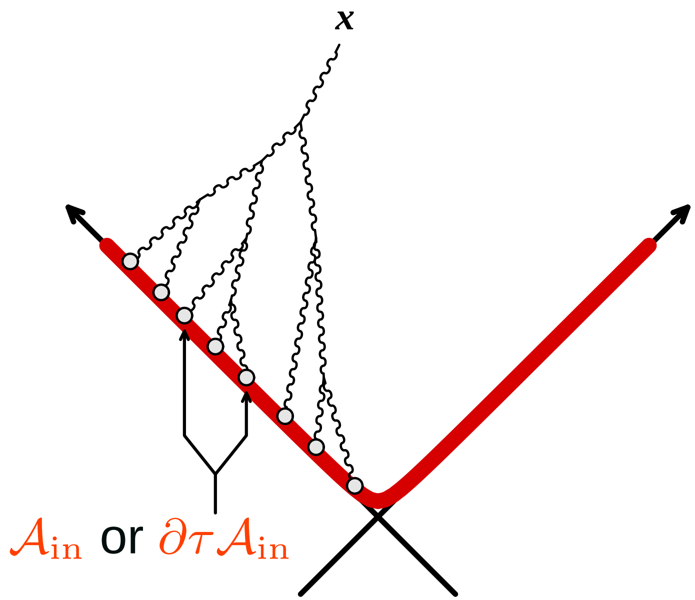
$$\triangleright \qquad \boldsymbol{T}_{\vec{\boldsymbol{u}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}}) \underset{\tau\to+\infty}{\sim} e^{\sqrt{\mu\tau}}$$

Action of Tu

- Green's formula for the retarded classical field $\mathcal{A}(x)$ above the light-cone :

$$\mathcal{A}(x) = \frac{g}{2}\int_{\mathrm{LC}+} d^4z\, G^0_R(x,z)\,\mathcal{A}^2(z) + \int_{\mathrm{LC}} d^3\vec{\boldsymbol{u}}\, G^0_R(x,u)\left[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\right]\mathcal{A}_{\mathrm{in}}(\vec{\boldsymbol{u}})$$

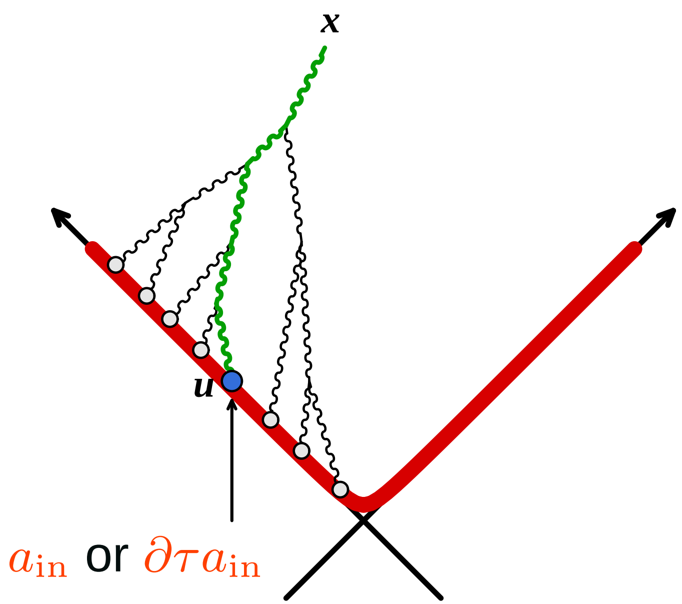
$\mathcal{A}(x) =$



Action of Tu

- Acting on $\mathcal{A}(x)$ with the operator $a_{\rm in}\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}$ replaces the initial $\mathcal{A}_{\rm in}$ or $\partial\tau\mathcal{A}_{\rm in}$ at point $\vec{\boldsymbol{u}}$ by $a_{\rm in}$ or $\partial\tau a_{\rm in}$ respectively :

$$\left[a_{\rm in}\cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right]\mathcal{A}(x) \quad =$$



- The line displayed in green can be seen as the retarded propagator of a fluctuation on top of the classical field between the point $\vec{\boldsymbol{u}}$ on the light-cone and the final point x